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Renovatio Monetae: When Gesell Taxes Worked*

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Abstract

Gesell taxes on money have recently received attention as a way of alleviating the zero lower bound on interest rates. Less known is that such taxes were an important method for generating seigniorage in medieval Europe for around two centuries. When a Gesell tax was levied, current coins ceased to be legal and had to be exchanged into new coins for a fee. This could occur as often as twice a year. Using a cash-in-advance model, we analyze under what conditions agents exchange coins and the tax generates revenues. A low exchange fee, high punishments for using old coins, and a long time period between re-mintings induce people to use new coins. We also analyze how prices fluctuated over an issue period.

Keywords: Seigniorage, Gesell tax, periodic re-coinage, cash-in-advance model

JEL classification: E42, E52, N13.

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1 Introduction

The idea of a tax on money holdings, first proposed by Gesell (1906), has received increasing attention in recent years due to the sudden empirical relevance of the zero lower bound. It is, however, less known that a (periodic) tax on money holdings existed for almost 200 years in large parts of medieval Europe, although the motivation for using the tax was different than today. Gesell taxes were implemented by coins being legal for only a limited period of time and, at the end of this period, they had to be exchanged for new coins for an ex ante known fee–an institution known as renovatio monetae or periodic re-coinage; see e.g. Allen (2012, p.35). Tax revenues depended not only on the fee charged but also on the duration of an issue. Both the exchange fee and the duration could vary across regions in the Middle Ages–a common annualized tax rate was 25 percent.

To generate revenues through seigniorage, the monetary authority benefits from creating an exchange monopoly for the currency. In a system with Gesell taxes and re-minting, in addition to competing with foreign coin issuers, the monetary authority competes with its own older issues. To limit the circulation of illegal coins, authorities penalized the use of invalid coins and required that fees, rents and fines be paid with current coins.

Although the disciplines of archaeology and numismatics have long been familiar with the presence of periodic re-coinage (Kluge, 2007, Allen, 2012, Svensson, 2016), evidence in written sources is scarce on the consequences of periodic re-coinage with respect to prices and people's usage of new and old coins. However, coin hoards indicate that old (illegal) coins often but not always circulated together with new coins; see Allen (2012, p. 520–23) and Haupt (1974, p. 29). In addition, written documents mention complaints against this monetary tax (Grinder-Hansen 2000, p. 51–52 and Hess, 2004, p. 19–20). Despite being common for an extended period of time, this type of monetary system has seldom if ever been analyzed theoretically in the economics or economic history literature.

The purpose of the present study is to fill this void in the literature. We formulate a cash-in-advance model in order to endogenize money demand, along the lines of Velde and Weber (2000) and Sargent and Smith (1997). An important reason for endogenizing money demand is that we can capture the implications of Gesell taxation in the form of periodic re-coinage on prices, seigniorage and people's decisions to use new or old coins for transactions in an economy. The model includes households, firms and a lord. To endogenize cash holdings, we introduce a non-cash alternative in the spirit of the cash and credit goods model of Lucas and Stokey (1987). In Svensson and Westermark (2016), we argue that the non-cash alternative can be interpreted as bartering. Credit is costly in the sense that it requires some labor input, along the lines of Khan, King, and Wolman (2003). Besides credit, households can hold both new and old coins, but only the new coins are legal in exchange. An issue of coins is only legal for a finite period of time; old coins must be re-minted at the re-coinage date to be considered legal in exchange. The lord charges a fee when there is a re-coinage so that for each old coin handed in, the household receives less than its full value in return. Despite being illegal, old coins can still be used for transactions. To deter the use of illegal coins, the lord's agents check whether legal means of payment are used in transactions. When they discover old coins, the coins are confiscated and re-minted into new coins. Thus, whether illegal coins circulate is endogenous in the model.

In the model, an interesting result is that Gesell taxes work when the period of time between two instances of re-coinage is sufficiently long. We also find that the system works when the exchange fee is sufficiently low and when the probability of being penalized for using old illegal coins is sufficiently high. Prices increase over time during an issue period and fall immediately after the re-coinage date, and, the higher the Gesell tax is, the higher the price increases are (as long as coins are handed in for re-coinage).

Empirical evidence indicates that periodic re-coinage ceased to be used after 150-200 years. To compare the periodic re-coinage system with a system of long-lived coins, we construct a model with long-lived coins in the spirit of Sussman and Zeira (2003). We find that increased fiscal spending tends to induce the lord to switch to systems with long-lived coins, since those systems can generate higher revenues. One alternative explanation for the switch to long-lived coins is an increase in the cost of non-cash alternatives, e.g., bartering. Interestingly, this makes periodic re-coinage more viable, since more transactions are made in the market, which leads to higher revenues for the lord. Thus, in light of the model, the switch to long-lived coins was driven by increased fiscal demands.

The paper is organized as follows. In section 2, we provide some stylized facts regarding medieval European coins and discuss the concept of and evidence for periodic re-coinage. Section 3 describes the model, and in section 4 we analyze the consequences of periodic re-coinage. Section 5 studies the choice between periodic re-coinage and long-lived coins and section 6 how the model fits the empirical evidence. In section 7 some of the assumptions in the model are discussed, and section 8 concludes.

2 The basics of medieval money and periodic re-coinage

Money in medieval Europe was overwhelmingly in the form of commodity money, based on silver;[1] fiat money did not exist in its pure form. As a regalian right, the right to mint belonged to the king/emperor. In addition to the right to determine, e.g., the design and the monetary standard, the coinage right encompassed the right to use the profits from minting and to decide which coins were legal; see Kluge (2007, p. 52). The right to mint for a region could be delegated, sold or pawned to other local authorities (local lords, laymen, churchmen, citizens) for a limited or unlimited period of time; see Kluge (2007, p. 53). The size of each currency area was usually smaller than today and could vary substantially. England was a single currency area (after 975), whereas Sweden and Denmark each had 2–3 areas. France and Germany had many small currency areas.

A commonly used monetary system in the Middle Ages was Gesell taxation in the form of periodic re-coinage. The main feature of such a re-coinage system is that coins circulate for a limited time, and, at the end of the period, the coins must be returned to the monetary authority and re-minted for an ex ante known fee, i.e., a Gesell tax. Thus, coins are "short-lived," in contrast to a "long-lived" monetary system in which the coins do not have a fixed period as a legal means of payment. According to written documents about periodic re-coinage, coins were usually exchanged on recurrent dates at a substantial fee and only valid for a limited time. The withdrawals were systematic and recurrent.

To obtain revenues from seigniorage, a coin issuer benefits from having an exchange monopoly in both long- and short-lived coinage systems. However, in a short-lived coinage system, the minting authority not only faces competition from other coin issuers but also from its own old issues that it minted. To create a monopoly position for its coins, laws stated that foreign coins were ipso facto invalid and had to be exchanged for the current local coins with the payment of an exchange fee in an amount determined by the coin

[1] The reason for this was the relative abundance of silver mines that led to a high supply of silver; see Spufford (1988, p.109ff, 119ff).

issuer.[2] Moreover, only one local coin type was considered legal at a given point in time.[3] To facilitate the verification of current and invalid coins, the main design of the coin was changed, whereas the monetary standard largely remained unchanged. This is similar to Gesell's original proposal, where stamps had to be attached to a bank note for it to retain its full value, which made it easy to verify whether the tax had been paid.

It may also be desirable to distinguish between periodic re-coinage and coinage reform, a distinction that has not necessarily been made explicit by historians and numismatists.[4]

2.1 Geographic extension of short-lived coinage systems

There is a substantial historical and numismatic literature that describes the extent of periodic re-coinage; see, e.g., Kluge (2007), Allen (2012), Bolton (2012) and Svensson (2016). Three methods have been used to identify periodic re-coinage and its frequency: written documents, the number of coin types per ruler and years, and the distribution of coin types in hoards (see Svensson (2016), appendix). There is a reasonable consensus in determining the extension of long- and short-lived coinage systems through time and space. Long-lived coins were common in northern Italy, France and Christian Spain from 900–1300. This system spread to England when the sterling was introduced during the second half of the 12th century. In France, in the 11th and 12th centuries, long-lived coins were dominant in the southern, western and central parts, and the rights to mint were distributed to many civil authorities. In northern Italy, long-lived coins likewise were dominant in the independent cities; see Kluge (2007, p. 136ff).

Short-lived coinage systems were the dominant monetary system in central, northern and eastern Europe from 1000–1300. The first periodic re-coinage in Europe occurred in Normandy between 930 and 1100 (Moesgaard 2015). Otherwise, a well-known example is England. Compared to Normandy, the English short-lived coins were valid in a large

[2] In 1231, the German king Henry VII (1222–35) published an edict in Worms stating that, in towns in Saxony with their own mints, goods could only be exchanged for coins from the local mint; see Mehl (2011, p. 33). However, when this edict was published, the system of coins constrained through time and space had been in force for a century in large parts of Germany.

[3] The coin issuer therefore has an incentive to ensure that foreign coins are not allowed to circulate. Moreover, to prevent illegal coins from circulating, the minting authority must control both the local market and the coinage; see Kluge (2007, p. 62–63).

[4] In fact, historians often use the term re-coinage for both periodic re-coinage and coinage reform. When a coinage reform is undertaken, coin validity is not constrained by time. A coinage reform also includes re-minting but is announced infrequently, and the validity period of the coins is not (explicitly) known in advance. Moreover, the coin and the monetary standard generally undergo considerable change.

currency area between 973 and 1125 (Spufford (1988, p. 92) and Bolton (2012, p. 87ff)).

The eastern parts of France and the western parts of Germany had periodic re-coinage in the 11th and 12th centuries; see Hess (2004, p. 19–20). However, the best examples of short-lived and geographically constrained coins can be found in central and eastern Germany and eastern Europe, where the currency areas were relatively small. Here, periodic re-coinage began in the middle of the 12th century and lasted until approximately 1300 and was especially frequent in areas where uni-faced bracteates were minted.[5]

Sweden had periodic re-coinage of bracteates in two of its three currency areas (especially in Svealand and to some extent in western Götaland) for more than a century, from 1180 to 1290; see Svensson (2015). Denmark introduced periodic re-coinage in the middle of the 12th century, which continued for 200 years with some interruptions; see Grinder-Hansen (2000, p. 61ff). Poland and Bohemia had periodic re-coinage in the 12$^{\text{th}}$ and 13$^{\text{th}}$ centuries; see Sejbal (1997, p. 26), Suchodolski (2012) and Vorel (2000, p. 341).

Empirical observations show that debasements in terms of lower weight or fineness occurred mainly in regions with long-lived coins (Kluge (2007), p. 64). For most regions with periodic re-coinage–England, Germany as well as eastern and northern Europe (see Table 1)–the silver fineness was sustained at a high level of at least 90 percent. Debasements only started in the 14th century when long-lived coins were introduced.[6]

2.2 Seigniorage and prices in systems with re-coinage

The seigniorage under re-coinage depends not only on the fee charged at the time of the re-coinage but also on the duration of an issue. Given the exchange fee and that money holdings are unaffected, the shorter the duration, the higher the revenues. Any reduction in money holdings due to shorter duration would reduce revenues.

There was a substantial variation in the level of seigniorage. In England from 973 to 1035, re-coinage occurred every sixth year. For approximately one century after 1035, English kings renewed their coinage every second or third year; see Spufford (1988, p. 92) and Bolton (2012, p. 99ff). The level of the fee is uncertain.[7]

[5] Bracteates are thin, uni-faced coins that were struck with only one die. A piece of soft material, such as leather or lead, was placed under the thin flan. Consequently, the design of the obverse can be seen as a mirror image on the reverse of the bracteates.

[6] An exception is Denmark in the period 1250–1350, when a civil war caused financial pressure, so that both periodic re-coinage and debasement were applied; see Grinder-Hansen (2000).

[7] According to Spufford (1988), four old coins were exchanged for three new coins, although this

Table 1: Exchange fees and duration of re-coinage in different areas

Region	Currency area♦	Period	Gesell tax★ (Annualized)	Duration years★	Method/Source†
Normandy	Small	930–1000	n.a.	3–5	2–3,
	Small	ca. 1000–1100	n.a.	1–3	Moesgaard (2015)
England	Large	973–1035	n.a.	6	1–3, Bolton (2012)
	Large	1035–1125	n.a.	2–3	2–3, Bolton (2012)
Germany, western✠	Small	ca. 1000–ca. 1300	mostly 25% (4.6%–25%)‡	1–5	1–3, Hess (2004)
Germany, eastern, northern✠	Small	ca. 1140–ca. 1330, sometimes until 15th cent.	mostly 25% (25%–44%)‡	$\frac{1}{2}$ or 1	1–3, Kluge (2007)
Teutonic Order in Prussia	Medium	1237–1364	17% (1.6%)	10	1–3, Paszkiewicz (2008)
Austria✠	Small	ca. 1200–ca. 1400	n.a.	1	2–3, Kluge (2007)
Denmark	Medium	ca. 1140–ca. 1330	33% (33%)	1, with inter-ruptions	1–3, Grinder-Hansen (2000)
Sweden, Svealand	Large	1180–1290	n.a.	1–5	2–3, Svensson
Sweden, Götaland	Large	1180–1290	n.a.	3–7	(2015)
Poland✠	Small	ca. 1100–ca. 1150	n.a.	3–7	1–3,
	Small	ca. 1150–ca. 1200	n.a.	1	Suchodolski
	Small	ca. 1200–ca. 1300	n.a.	$\frac{1}{3}$ or $\frac{1}{2}$	(2012)
Bohemia-Moravia	Medium	ca. 1150–1225	n.a.	1	Sejbal (1997) and
	Medium	1225–ca. 1300	n.a.	$\frac{1}{2}$	Vorel (2000)

Notes: ♦ We do not use a formal definition of area size. By a large area, we mean a country or a substantial part of a country, such as England or Svealand. A small area is usually a city and its hinterland. A medium-sized area is somewhere in-between and is exemplified by the kingdom of Wessex. †Methods: 1) Written sources; 2) No. of types per time period; 3) Distribution of coin hoards. ✠ Various mints and authorities. ‡Annualized rate based on a fee of 25 percent. ★ When known.

In other areas in Europe, the duration was often significantly shorter. Austria and Brandenburg had annual re-coinage until the end of the 14th century and 1369, respectively (Kluge (2007, p. 108, 119)). Some German mints had biannual or annual renewals until the 14th or 15th centuries (e.g., Brunswick until 1412); see Kluge (2007, p. 105). In Denmark, re-coinage was mostly annual; see Grinder-Hansen (2000, p. 61ff). In Poland, King Boleslaw (1102–38) began with irregular re-coinages–every third to seventh year–but later the frequency increased. In the late 12th century, coin renewals were annual, and in the 13th century, they occurred two or three times per year; see Suchodolski (2012). Bohemia also had re-coinage at least once each year in the 12th and 13th centuries; see

calculation is based on a rather uncertain weight analysis. If the gross seigniorage was 25 percent every sixth year, the annualized rate was almost 4 percent.

Sejbal (1997, p. 83) and Vorel (2000, p. 26). In contrast, the Teutonic Order had periodic re-coinages only every tenth year between 1237 and 1364; see Paszkiewicz (2008).

The exchange fee in Germany was generally four old coins for three new coins, i.e., a Gesell tax of 25 percent; see Svensson (2016, p. 1114). In Denmark, the tax–three old coins for two new coins–was higher, at 33 percent; see Grinder-Hansen (2000, p. 179). The annualized tax in Germany could be very high–up to 44 percent.[8] The Teutonic Order in Prussia had a relatively low exchange fee of seven old coins for six new coins, a tax rate of almost 17 percent, or in annualized terms, 1.6 percent; see Paszkiewicz (2008).

2.3 Success, monitoring and enforcement of re-coinage

There was considerable variation in the success of re-coinage. The coin hoards discovered to date can tell us a great deal about the success of re-coinage. In Germany, taxation was high and re-coinage occurred frequently; see Table 1. Unsurprisingly, hoards in Germany from this period (1100–1300) usually contain many different issues of local coins as well as many foreign coins, i.e., locally invalid coins; see Svensson (2016), Table 3. This indicates that the authorities had problems enforcing circulation of their coins. By avoiding some coin renewals and saving their retired coins, people could accumulate silver or use old coins illegally. In contrast, hoard evidence from England indicates that the periodic re-coinage systems were partly successful; see Dolley (1983). Almost all of the coins in hoards are of the last type during the period 973–1035, when coins were exchanged every sixth year; see Table 2. However, from 1035 to 1125, only slightly more than half of the coins were of the last type, indicating that the system worked well up to 1035 but less so after that. One reason may be that the seigniorage for the later period was higher because of the shorter period of time between withdrawals (at an unchanged exchange fee).

Because hoards often contain illegal coins, the incentives to try to avoid re-coinage fees appear to occasionally have been rather high. To curb the circulation of illegal coins, monetary authorities used different methods to control the usage of coins. The usage of invalid coins was deemed illegal and penalized, although the possession of invalid coins was mostly legal.[9] If an inhabitant used foreign coins or old local coins for transactions and

[8] The annualized rate is based on a bi-annual tax of 25 percent as in Magdeburg (Mehl (2011, p. 85)).
[9] City laws in Germany stated that neither the mint master nor a judge was allowed to enter homes and search for invalid coins (Haupt 1974, p. 29).

Table 2: The composition of English coin hoards 979–1125. Number of coin hoards, number of coins and shares

Period		973–1035		1035–1125	
Years between re-coinages		6 years		2–3 years	
		No. of coins	Share	No. of coins	Share
Coins from	Last issue	886	86.5%	8 771	54.3%
	Second to last issue	137	13.4%	1 724	10.7%
	Third to last issue	1	0.1%	698	4.3%
	Earlier issues	0	0.0%	4 964	30.7%
Total number of coins		1 024	100.0%	16 157	100.0%

Notes: Source Svensson (2016), Table 2.

was detected, the penalty could be severe. Moreover, sheriffs and other administrators who accepted taxes or fees in invalid coins were penalized; see Haupt (1974, p. 29), Grinder-Hansen (2000, p. 69), and Hess (2004, p. 16). Controlling the usage of current coins was likely easier in cities than in the countryside.[10] The minting authority could also indirectly control the coin circulation by requiring that fees, rents and fines were to be paid with current coins; see Grinder-Hansen (2000, p. 69) and Hess (2004, p. 19).

3 The economic environment

In this section, we outline a model of periodic re-coinage. The economy consists of households, firms and a lord. There are trade opportunities with the rest of the world, and goods can be exchanged for silver on the world market at a fixed world market relative price γ. We endogenize cash holdings by assuming that households care about consumption of two types of goods, a cash good c_{1t} and a credit good c_{2t}. Total consumption is $c_t = c_{1t} + c_{2t}$. Households can trade the cash good by using coins on the market, facing a cash-in-advance constraint. The credit good can be paid for with loans. All loans are settled within a time period. Household money holdings consist of new and old coins, m_t^n

[10]Irrespective of the size of the currency area, systems with short-lived coins could often be strictly enforced only in a limited area of the authority's domain, such as in cities. If most trade occurred in cities, this restriction may not be a strong constraint, however. Normally, the city border demarcated the area that included the jurisdiction of the city in the Middle Ages. The use of foreign and retired local coins within the city border was forbidden. This state of affairs is well documented in an 1188 letter from Emperor Friedrich I (1152–90) to the Bishop of Merseburg (Thuringia) regarding an extension of the city. The document plainly states that the market area boundary includes the entire city, not just the physical marketplaces; see Hess (2004, p. 16). A document from Erfurt (1248/51) shows that only current local coins could be used for transactions in the town, whereas retired local coins and foreign coins were allowed for transactions outside of the city border; see Hess (2004, p. 16).

and m_t^o, made of silver.[11] Only new coins are legal in exchange, but households can use both types in transactions. Thus, whether illegal (old) coins circulate is endogenous in the model. The new coins are withdrawn from circulation every Tth period. Specifically, to be considered legal in exchange after a withdrawal, coins must be handed in to be re-minted. Any coin that is not re-minted is not legal after the re-coinage date and subject to the risk of confiscation when used in transactions, i.e., treated as an old (illegal) coin. The lord charges a Gesell tax τ at the time of each withdrawal. Then, for each coin handed in for re-minting, the household receives $1 - \tau$ new coins in return. Although old coins can be used for transactions, it is costly to do so since they can be confiscated. Specifically, lord agents monitor each cash transaction with some probability and check whether the legal means of payment is used. If they discover old coins, the coins are confiscated, re-minted as new coins and used to fund the lord's expenditures. The probability is assumed to be decreasing in the total number of transactions monitored, c_{1t}^{agg}, and is given by $1 - \chi\left(c_{1t}^{agg}\right)$.[12] Because the lord's agents confiscate old coins, old and new coins do not need to circulate at par, and e_t denotes the exchange rate between old and new coins.

The firm can melt (mint) coins and export (import) silver in exchange for the consumption goods. The lord's revenues, i.e., from minting, re-minting and confiscations, are spent on the lord's consumption, g_t. At the beginning of a period t, households have an endowment of goods ξ_t and a stock of new and old coins. The household endowment of goods is sold to the firms in return for a claim on firm profits. Then, competitive firms decide whether to produce: 1) two consumption goods c_{1t} and c_{2t}, using the endowment or by exporting silver through melting of new (old) coins, μ_t^n (μ_t^o); and 2) minting n_t^n new coins by importing silver or melting old coins.[13] Shopping begins with households buying consumption goods from firms at competitively determined prices p_t. As in Lucas and Stokey (1987), the prices on cash and credit goods are the same. If coins are minted, firms pay the same fee as when coins are returned on the re-coinage date. Then, the profits are returned to the households in the form of dividends. Finally, on the re-coinage date,

[11] The amount of silver is identical in old and new coins. Also, for simplicity, we ignore foreign coins.

[12] Note that, if the household uses more illegal coins in transactions, then more of these coins will be confiscated; the amount confiscated is $\left(1 - \chi\left(c_{1t}^{agg}\right)\right) m_t^o$.

[13] A motivation for competitive mints is that, e.g., in the 11th–12th centuries, England had up to approximately 70 active mints at times; see Allen (2012, p. 16 and p. 42f). Moreover, these mints were sometimes farmed out; see Allen (2012, p. 9).

households hand in r_t^h coins to the firm for re-minting into new coins.

3.1 The firm

During each period, the firm sells c_t and g_t and mints and melts coins. Due to the Gesell tax, new and old coins are valued differently at the re-coinage date. Letting q_t denote the price of new coins in terms of old, the value of an old coin in terms of new is $\frac{1}{q_t}$. Firm profits are then, measured in new coins,

$$\Pi_t = p_t (c_t + g_t) + (1 - \tau) n_t^n - \mu_t^n - e_t \mu_t^o + (1 - \tau) n_t^r - \frac{1}{q_t} r_t. \tag{1}$$

Here, n_t^r is the amount of new re-coined coins and r_t the amount of old coins handed in for re-coinage. Mintage and melting must be non-negative, and, hence, the firm faces the following constraints related to mintage and melting: $n_t^n \geq 0$, $\mu_t^n \geq 0$ and $\mu_t^o \geq 0$. The firm maximizes its profits in (1) subject to these constraints and

$$c_t + g_t \leq \xi_t + \text{Im}_t, \tag{2}$$

where Im_t is imports of goods. Let $\hat{b}$ be the grams of silver in a coin. Then

$$\text{Im}_t = \frac{\hat{b}}{\gamma} (\mu_t^n + \mu_t^o - n_t^n), \tag{3}$$

where γ is the relative world market price of silver. We normalize $b = \frac{\hat{b}}{\gamma}$ to one.

The firm's decision whether to export or import goods in exchange for silver determines mintage and melting of new and old coins. From the firm's first-order condition for minting, if $1 - \tau > p_t$ then $n_t^n = \infty$, if $1 - \tau < p_t$ then $n_t^n = 0$, and if

$$1 - \tau = p_t \text{ then } n_t^n \in [0, \infty). \tag{4}$$

Thus, if p_t is high relative to the world market price of silver, i.e., $p_t > 1 - \tau$, it is unprofitable to export goods for silver on the world market, implying that mintage is zero. If p_t is low, i.e., $p_t < 1 - \tau$, then the firm makes a positive profit on each new coin that it mints. Equilibrium then requires that $1 - \tau \leq p_t$ with equality, whenever $n_t^n > 0$.

The firm decision to import goods in exchange for silver leads to the following conditions for the melting of new coins: if $p_t > 1$ then $\mu_t^n = \infty$, if $p_t < 1$ then $\mu_t^n = 0$, and if

$$p_t = 1 \text{ then } \mu_t^n \in [0, \infty). \tag{5}$$

Hence, if the price of the goods is low, i.e., $p_t < 1$, it is not profitable for the firm to melt coins and transform them into goods through exports. If the price is higher than 1, the firm makes a positive profit on each new coin that it melts. Repeating the same for μ_t^o gives the following: if $p_t > e_t$ then $\mu_t^o = \infty$, if $p_t < e_t$ then $\mu_t^o = 0$, and if

$$p_t = e_t \text{ then } \mu_t^o \in [0, \infty). \tag{6}$$

Finally, noting that $n_t^r = r_t$, the first-order condition regarding re-coinage is, if $q_t < \frac{1}{1-\tau}$ then $n_t^r = \infty$, if $q_t > \frac{1}{1-\tau}$ then $n_t^r = 0$, and if

$$q_t = \frac{1}{1-\tau} \text{ then } n_t^r \in [0, \infty). \tag{7}$$

3.2 The household

The household preferences are[14]

$$\sum_{t=0}^{\infty} \beta^t \left[u\left(c_t\right) - v\left(c_{2t}\right) \right], \tag{8}$$

where $c_t = c_{1t} + c_{2t}$. One way of interpreting v is that it is costly (in terms of labor) to use credit, along the lines of Khan, King, and Wolman (2003). Then $v\left(c_{2t}\right)$ is the disutility of labor from buying c_{2t} of the credit good. In Svensson and Westermark (2016), we argue that this formulation can be interpreted in terms of bartering, where the credit good is traded via bartering, which is costly in terms of labor. We assume that u (v) is strictly increasing and strictly concave (convex). We impose the standard Inada condition so that $\lim_{c \to 0} u'\left(c\right) \to \infty$. Also, $\lim_{c_2 \to 0} v'\left(c_2\right) = 0$. Following Velde and Weber (2000), the endowment is transferred to firms in return for a claim on profits. The household maximizes utility in (8), subject to the CIA and budget constraints

$$p_t c_{1t} = m_t^n + e_t \chi\left(c_{1t}^{agg}\right) m_t^o, \tag{9}$$

[14]In terms of Lucas and Stokey (1987), $u\left(c_{1t}, c_{2t}\right) = u\left(c_t\right) - v\left(c_{2t}\right)$.

$$
\begin{aligned}
\left((1-\mathbb{I}_t)+\mathbb{I}_t q_t\right) m_{t+1}^n + e_t m_{t+1}^o \quad \leq \quad & (1-\mathbb{I}_t)\,\Pi_t^n + \mathbb{I}_t r_t^h + e_t\left(\Pi_t^o + \mathbb{I}_t\left(\Pi_t^n - r_t^h\right)\right) \\
& + m_t^n + e_t \chi\left(c_{1t}^{agg}\right) m_t^o - p_t c_{1t} - p_t c_{2t},
\end{aligned} \tag{10}
$$

where $\mathbb{I}_t = 1$ if $t = T, 2T, 3T$ and 0 otherwise, Π_t^n are firm dividends in new coins, and Π_t^o dividends in old coins. Also, $c_t \geq 0$, $m_{t+1}^n \geq 0$, and $m_{t+1}^o \geq 0$. Furthermore, $r_t^h \in [0, \Pi_t^n]$ if $\mathbb{I}_t = 1$ and $r_t^h = 0$ otherwise.

Here, we describe the household optimality conditions, assuming $c_t > 0$ and $p_t > 0$ for all t, which holds in equilibrium. Whether old or new coins are held depends on how exchange rates affect their relative return. Using the first-order conditions with respect to c_t and m_{t+1}^n, if $m_{t+1}^o > 0$ then

$$
\left((1-\mathbb{I}_t)+\mathbb{I}_t q_t\right) e_{t+1} \chi\left(c_{1t+1}^{agg}\right) \geq e_t. \tag{11}
$$

Since the consumer holds old coins in period $t+1$, the exchange rates in periods t and $t+1$ have to give the consumer incentives not to only hold new coins. Then, it follows that the exchange rate has to increase by at least $1/\chi\left(c_{1t+1}^{agg}\right)$ between adjacent periods, except in the withdrawal period when it appreciates by $1/q_t\chi\left(c_{1t+1}^{agg}\right)$. The appreciation of the exchange rates compensates the consumer for the loss due to confiscations so that the consumer does not lose in value terms by holding an old coin instead of a new.

If $m_{t+1}^n > 0$ then

$$
\left((1-\mathbb{I}_t)+\mathbb{I}_t q_t\right) e_{t+1} \chi\left(c_{1t+1}^{agg}\right) \leq e_t. \tag{12}
$$

Since the consumer now holds new coins in period $t+1$, the exchange rates in period t and $t+1$ have to give the consumer incentives to not only hold old coins, implying that the exchange rate increase is bounded above by $1/\left((1-\mathbb{I}_t)+\mathbb{I}_t q_t\right)\chi\left(c_{1t}^{agg}\right)$.

Finally, the household optimally chooses the share of coins handed in for re-coinage, r_t^h in periods $t = T, 2T$, etc.; if $e_t < 1$ then $r_t^h = \infty$, if $e_t > 1$ then $r_t^h = 0$, and if

$$
e_t = 1 \text{ then } r_t^h \in [0, \infty). \tag{13}
$$

When choosing how to allocate the new coins in period T to new and old coins in the next period, the household takes into account the coins' relative value. When handing in a coin for re-minting, the value is one. When not handing it in, the value is e_t. Thus, if

$e_t < 1$, all new coins are re-minted, and, if $e_t > 1$, no new coins are re-minted.

By using the first-order condition with respect to c_{1t}, c_{2t} and m_t^n, we have, when $t-1 \neq T$ and $m_t^n > 0$,

$$\frac{p_t}{p_{t-1}} = \beta \frac{u'(c_t)}{u'(c_{t-1}) - v'(c_{2t-1})} \tag{14}$$

and, when $t-1 = T$ and $m_t^n > 0$,

$$\frac{p_t}{p_{t-1}} = \beta (1-\tau) \frac{u'(c_t)}{u'(c_{t-1}) - v'(c_{2t-1})}. \tag{15}$$

When households optimally choose nominal money holdings in the case when $t-1 \neq T$, the payoff gain in period t of increasing m_t^n is $\beta u'(c_t)/p_t$ and the payoff loss in period $t-1$ is $(u'(c_{t-1}) - v'(c_{2t-1}))/p_{t-1}$. Equating these yields (14). When old coins are held ($m_t^o > 0$), we get, using the first-order conditions with respect to c_{1t}, c_{2t} and m_t^o,

$$\frac{p_t}{p_{t-1}} = \frac{\beta e_t \chi\left(c_{1t-1}^{agg}\right)}{e_{t-1}} \frac{u'(c_t)}{u'(c_{t-1}) - v'(c_{2t-1})}. \tag{16}$$

3.3 The lord

The lord gets revenue from coin withdrawals and confiscation of illegal coins. The lord hands in all confiscated old coins to be minted into new coins. Letting $m_t^L \geq 0$ denote coins stored by the lord, the lord budget constraint is

$$m_{t+1}^L = \tau\left(n_t^n + r_t^L + \mathbb{I}_t r_t^h\right) + \frac{1}{q_t} r_t^L + (1-\mathbb{I}_t) m_t^L - p_t g_t, \tag{17}$$

where $r_t^L = (1 - \chi(c_{1t}^{agg})) m_t^o + \mathbb{I}_t m_t^L$. Thus, the lord uses revenues from money withdrawals through r_t^h, from new mintage through n_t^n, confiscations through m_t^o and previously stored coins m_t^L to spend on consumption (g_t) and coins stored to the next period m_{t+1}^L. In order to simplify the derivation of the results, we restrict g_t to be constant over time.

3.4 Money transition and resource constraints

When trading cash goods, households spend $m_t^n + e_t \chi(c_{1t}^{agg}) m_t^o$ on goods and the government $p_t g_t$, which is equal to firm profits. Hence, reimbursment to households of new coins after trading is $\Pi_t^n = m_t^n + p_t g_t + (1-\tau) n_t^n - \mu_t^n$ and of old coins $e_t \chi(c_{1t}^{agg}) m_t^o - e_t \mu_t^o$. Then, the household stocks of new and old coins evolve according to, using (10) and that

r_t^h coins handed in for re-coinage gives $\frac{1}{q_t} r_t^h = (1-\tau) r_t^h$ new coins in return,

$$\begin{aligned} m_{t+1}^n &= (1-\mathbb{I}_t)(m_t^n + p_t g_t + (1-\tau) n_t^n - \mu_t^n) + \mathbb{I}_t (1-\tau) r_t^h \qquad (18) \\ m_{t+1}^o &= \chi(c_{1t}^{agg}) m_t^o - \mu_t^o + \mathbb{I}_t \left(m_t^n + p_t g_t + (1-\tau) n_t^n - \mu_t^n - r_t^h\right). \qquad (19) \end{aligned}$$

We also have the re-coinage constraint $r_t = r_t^h + r_t^L$.

By symmetry, we have $c_{1t}^{agg} = c_{1t}$. Finally, we have the goods' market clearing constraint

$$c_{1t} + c_{2t} + g_t = \xi_t + \mathrm{Im}_t. \qquad (20)$$

4 Equilibria

We now proceed to analyze equilibria of the above model.

Definition 1 *An equilibrium is a collection* $\{m_{t+1}^n\}$, $\{m_{t+1}^o\}$, $\{m_{t+1}^L\}$, $\{n_t^n\}$, $\{\mu_t^n\}$, $\{\mu_t^o\}$, $\{n_t^L\}$, $\{\mu_t^L\}$, $\{c_{1t}\}$, $\{c_{2t}\}$, $\{g_t\}$, $\{\mathrm{Im}_t\}$, $\{r_t^h\}$, $\{r_t^L\}$, $\{p_t\}$, $\{q_t\}$, *and* $\{e_t\}$ *such that i) the household maximizes (8) subject to (9), (10),* $r_t^h \in [0, \Pi_t^n]$ *when* $\mathbb{I}_t = 1$ *and* $r_t^h = 0$ *otherwise and the boundary constraints; ii) the firm maximizes (1) subject to its boundary constraints and (2); iii) that (17), (18), (19),* $r_t = r_t^h + r_t^L$, *and (20) hold.*

For the rest of the analysis, we assume that the endowment is constant; $\xi_t = \xi$. For the lord, the budget is balanced over the cycle. Thus, summing (17) over the cycle,

$$\sum_{t=1}^{T} p_t g_t = \tau r_T^h + \tau \sum_{t=1}^{T} n_t^n + \sum_{t=1}^{T} (1 - \chi(c_{1t}^{agg})) m_t^o. \qquad (21)$$

Note that due to the fact that money withdrawals occur infrequently, i.e., every Tth period, a steady state cannot be expected to exist. Therefore, we instead restrict the attention to *cyclical equilibria.* Thus, consider an issue with length T where an issue starts just after a withdrawal and ends just before the next withdrawal. Let $L_r^T = \{\tilde{r} : \tilde{r} = nT + r$ for $n \in N^+\}$ denote all time periods corresponding to a given period r in some issue.

Definition 2 *Given that money withdrawals occur every Tth period, an equilibrium is said to be* ***cyclical*** *if it satisfies* $m_{\hat{r}}^n = m_{\bar{r}}^n$, $m_{\hat{r}}^o = m_{\bar{r}}^o$, $m_{\hat{r}}^L = m_{\bar{r}}^L$, $n_{\hat{r}}^n = n_{\bar{r}}^n$, $\mu_{\hat{r}}^n = \mu_{\bar{r}}^n$, $\mu_{\hat{r}}^o = \mu_{\bar{r}}^o$, $c_{1\hat{r}} = c_{1\bar{r}}$, $c_{2\hat{r}} = c_{2\bar{r}}$, $\mathrm{Im}_{\hat{r}} = \mathrm{Im}_{\bar{r}}$, $r_{\hat{r}}^h = r_{\bar{r}}^h$, $r_{\hat{r}}^L = r_{\bar{r}}^L$, $p_{\hat{r}} = p_{\bar{r}}$, *and* $e_{\hat{r}} = e_{\bar{r}}$ *for all* $r \in \{1, \ldots, T\}$ *such that* $\hat{r}, \bar{r} \in L_r^T$.

The definition of cyclicality requires that, at the same point in two different issues, the variables attain the same value, i.e., for example $m^n_{\hat{r}} = m^n_{\bar{r}}$.

We use the below example (where there is a withdrawal of coins every second period) to describe the derivation of and intuition behind many of the results in the section. All proofs in the general case are relegated to the appendix.

Example 1 *Only new coins are held in equilibrium, $T = 2$. For simplicity, we set $m^L_1 = 0$. We now show that minting is zero in equilibrium. First, suppose that melting (minting) is positive in period 1 (2), i.e., $\mu^n_1 > 0$ and $n^n_2 > 0$, and, hence, $\mathrm{Im}_1 > 0$ and $\mathrm{Im}_2 < 0$. From firm optimization, prices are $p_1 = 1$ and $p_2 = 1 - \tau$. The constraints on household choices also impose conditions on household consumption of cash and credit goods. Specifically, using the definition of imports, the CIA constraint (9), the resource constraint (2), and money transition (18), we can derive the following (quantity theory-related) expressions*

$$\begin{aligned} p_1 (c_{11} + g - \mathrm{Im}_1) &= p_1 (\xi - c_{21}) = m^n_2 \\ p_2 (c_{12} + g - \mathrm{Im}_2) &= p_2 (\xi - c_{22}) = \frac{1}{1-\tau} m^n_1. \end{aligned} \tag{22}$$

Since goods prices are high in period 1 and low in period 2, credit good consumption is low in period 1 and high in period 2, i.e., since (18) implies $m^n_1 > (1-\tau) m^n_2$ and using (22), we have $c_{22} < c_{21}$. Moreover, since goods are imported (exported) in period 1 (2), we have $c_1 > c_2$. Also, since households consume more in period 1 than 2 of both aggregate and credit goods, the effect on the payoff in period 2 of an increase in m^n_2 is relatively high. The payoff gain in period 2 of increasing m^n_2 is $\beta u' (c_2) / p_2$, and the payoff loss in period 1 is $(u'(c_1) - v'(c_{21})) / p_1$. Prices adjust so that these are equal and (14) - (15) hold, and, hence, goods' prices must be lower in period 1 than in period 2. Then, firm and household behavior are inconsistent since $p_2 = (1-\tau) p_1$ from firm optimization, a contradiction. When $\mathrm{Im}_1 < 0$ and $\mathrm{Im}_2 > 0$, a similar argument establishes a contradiction.[15] *Hence, imports, minting and melting are zero for $t = 1, 2$. Since aggregate consumption is constant over the cycle, cash and credit good consumption are*

[15] *Along the lines of the first case, we can establish that $c_2 > c_1$ and $c_{21} < c_{22}$. Hence, $v'(c_{22}) > v'(c_{21})$, implying $u'(c_2) - v'(c_{22}) < u'(c_1) - v'(c_{21})$. Then, (14) and (15) establish a contradiction:*

$$\frac{1}{1-\tau} \frac{p_1}{p_2} = 1 > \frac{p_2}{p_1} = \frac{1}{1-\tau}. \tag{23}$$

also constant.[16]

We now proceed to analyze properties of equilibria. The following Lemma states that the above results holds in the general case, i.e., imports are zero in a cyclical equilibrium.

Lemma 1 *When only new coins are held, imports are zero,* $\mathrm{Im}_t = 0$ *for all* t.

We also have the following corollary that generalizes equilibrium consumption choices.

Corollary 1 *When only new coins are held, total consumption,* c_t*, and the amount of consumption goods bought using cash,* c_{1t}*, and credit,* c_{2t}*, is constant over the cycle.*

Thus, from Lemma 1 and the Corollary above, imports are zero and consumption is constant over the cycle. The (quantity theory-related) result in expression (22) can be shown to hold generally. By using (18) in (9), we can derive the following Lemma.

Lemma 2 *The CIA constraint (9) is, when* $t \neq T$,

$$p_t\left(\xi - c_{2t}\right) = m_{t+1}^n + e_t m_{t+1}^o \tag{24}$$

and, when $t = T$ *and* $r_t^h > 0$,

$$p_t\left(\xi - c_{2t}\right) = \frac{1}{1-\tau} m_{t+1}^n + e_t m_{t+1}^o \tag{25}$$

and, when $t = T$ *and* $r_t^h = 0$,

$$p_t\left(\xi - c_{2t}\right) = \left(1 - e_t\right)\left(m_t^n + p_t g + \left(1-\tau\right) n_t^n - \mu_t^n\right) + e_t m_{t+1}^o. \tag{26}$$

Example 1, continued. *We now describe equilibrium prices. From above, imports are zero and consumption* is *constant over the cycle* ($c_{11} = c_{12} = \bar{c}_1$ *and* $c_{21} = c_{22} = \bar{c}_2$). *Money holdings increase by* $p_1 g$ *at the end of period* 1 *and decrease due to the tax at the*

[16] *Using (22),* $m_2^n = m_1^n + p_1 g$ *and (14), letting* $\bar{c} = c_1 = c_2$*, we have* $v'\left(c_{21}\right) = u'\left(\bar{c}\right)\left(1 - \beta\frac{\xi - c_{22} - g}{\xi - c_{21}}\right)$ *and* $v'\left(c_{22}\right) = u'\left(\bar{c}\right)\left(1 - \beta\frac{\xi - c_{21} - g}{\xi - c_{22}}\right)$*. If* $c_{22} > c_{21}$*, then* $v'\left(c_{22}\right) > v'\left(c_{21}\right)$*. Also,* $\frac{\xi - c_{22} - g}{\xi - c_{21}} < \frac{\xi - c_{21} - g}{\xi - c_{22}}$*, and, hence,* $v'\left(c_{22}\right) < v'\left(c_{21}\right)$*, a contradiction. A similar argument rules out* $c_{21} > c_{22}$*, and, hence,* $c_{21} = c_{22}$*.*

end of period 2. Using the CIA constraint (22) and money transition (18), $m_2^n = \frac{\bar{c}_1+g}{\bar{c}_1} m_1^n$ and $m_1^n = (1-\tau)\frac{\bar{c}_1+g}{\bar{c}_1} m_2^n$*. Hence,* $\frac{\bar{c}_1}{\bar{c}_1+g} = \sqrt{1-\tau}$*. Then, using (22), we have*

$$p_2 = \frac{1}{\sqrt{1-\tau}} p_1, \tag{27}$$

i.e., *prices increase by* $\frac{1}{\sqrt{1-\tau}}$ *between periods* 1 *and* 2*. Since* $p_2 \leq 1$ *from firm optimization, any combination of prices with* $p_2 = \frac{1}{\sqrt{1-\tau}} p_1$ *where* $p_1 \in [1-\tau, \sqrt{1-\tau}]$ *is feasible. Each such price is associated with a unique level of money holdings via the CIA constraint.*[17] *Finally, consider exchange rate restrictions for the equilibrium. Let the constant retention rate when holding old coins be denoted* $\bar{\chi} = \chi(\bar{c}_1)$*. Since households hold only new coins, for it to be profitable for the firm to re-coin we must have* $q_t \geq \frac{1}{1-\tau}$*. For households to choose to hold only new coins, see (12), the value of old coins cannot appreciate too much, i.e.,* $e_2\bar{\chi} \leq e_1$ and $e_1\bar{\chi} \leq (1-\tau)e_2$*, and since households choose to re-coin, old coins cannot be worth too much at the re-coinage date, i.e.,* $e_2 \leq 1$*. Combining gives the following requirement for households to hold only new coins in equilibrium;*

$$1-\tau \geq \bar{\chi}^2. \tag{28}$$

In general, prices grow over time, except at the re-coinage date, due to household money holdings increasing by $p_t g$ from lord spending. When old coins are also held, the price increase is similar to Example 1. Specifically, using the CIA constraint, Lemma 2 and, in the case when old coins are held, that $e_t\bar{\chi} = e_{t-1}$ and $m_t^o = \bar{\chi} m_{t-1}^o$, we have,

$$\frac{p_t}{p_{t-1}} = \frac{\bar{c}_1 + g}{\bar{c}_1}. \tag{29}$$

We have the following theorem.

Theorem 1 *An equilibrium where only new coins are held exists if* $1-\tau > \bar{\chi}^T$ *where* $\bar{\chi} = \chi(\bar{c}_1)$*. In equilibrium,* $n_t^n = \mu_t^n = 0$ *for all* t *and prices increase at the rate* $(1-\tau)^{-\frac{1}{T}}$ *during an issue and drop between periods* T *and* $T+1$*.*

Since imports, minting and melting are zero and cash and credit good consumption are constant over the cycle when only new coins are held, we restrict attention to such

[17] Note that $\bar{c}_1$, $\bar{c}_2$ and g are determined from (14), the lord budget constraint (17) and the market clearing constraint (20). For details on how to solve for money holdings, see the proof of Theorem 1.

equilibria when old coins are also held. Note that the equilibrium where both old and new coins are held is generic. The issue regarding non-generic equilibria is related to which coins are *handed in* for re-coinage. There is a non-generic equilibrium where some but not all legal coins are handed in (when $1-\tau=\bar{\chi}^{T}$). When not all coins are re-coined, households hold both old and new coins since the lord re-coins old coins and then spends them on g_t. Then, firm profits partly consists of new coins, which are disbursed to the households.

Theorem 2 *Suppose old coins are held. A cyclical equilibrium where imports, minting and melting are zero and cash and credit good consumption are constant over the cycle exists when $1-\tau\leq\bar{\chi}^{T}$, where $\bar{\chi}=\chi(\bar{c}_1)$. In any equilibrium, prices increase by the rate in (29) during an issue and drop between periods T and $T+1$. If $1-\tau<\bar{\chi}^{T}$, no coins are handed in for re-coinage and prices increase at the rate $\frac{1}{\bar{\chi}}$ during an issue.*

The results for increasing prices in equilibria where only new coins are held follow from the fact that government spending implies that household money holdings increase over the cycle.[18] As long as only new coins are held, price increases are higher the higher the Gesell tax since a higher tax leads to larger government spending and, in turn, a greater increase in household money holdings during a cycle. When $1-\tau<\bar{\chi}^{T}$ so that old coins are also held, price increases depend on $\bar{\chi}$. Because no coins are handed in for re-coinage, the only source of government revenues is the confiscation of illegal coins, and thus $\bar{\chi}$ determines government spending and how household money holdings evolve.[19]

The cutoff values for whether old coins are held depend on $\bar{\chi}$ and τ. The intuition behind this cutoff is that, assuming that households want to hold both new and old coins, the exchange rate must appreciate at a rate of one over the confiscation rate $\bar{\chi}$ (using (11) and (12)), i.e., $1/\bar{\chi}$, when there is no re-coinage and at rate $\frac{1}{\bar{\chi}q_t}$ at the re-coinage date, due to the change in relative price of old and new coins. We have $e_1=\bar{\chi}e_2=\cdots=\bar{\chi}^{T-1}e_T$. Since not all new coins are handed in for re-coinage, households must weakly prefer not to hand in new coins, and, hence, $e_1\bar{\chi}\geq(1-\tau)\,e_T$. Thus, $1-\tau\leq\bar{\chi}^{T}$.

[18] Government spending increases firm profits, which then are disbursed to households.

[19] Note that the value of old coins is indeterminate in equilibrium; see the proof for details. Hence, the price level is also indeterminate as it depends on the exchange rate; see (9). This in turn implies that government spending depends on the exchange rate and that spending is highest when the exchange rate is at its lowest possible level, i.e., $e_T=1$. If this is the case, prices grow by $\bar{\chi}$. Otherwise, the growth rate is lower because the increase in private sector money holdings over the cycle is lower; see (18).

4.1 Welfare, taxes and spending

We now analyze the effect of taxes (and frequency of re-coinages) on household welfare and lord consumption. When only new coins are used, the equilibrium is given by (14), (20) and (29). Using that we have $p_t/p_{t-1} = (1-\tau)^{-\frac{1}{T}}$, consumption and spending depend on $\hat{T} \equiv (1-\tau)^{-\frac{1}{T}}$. Hence, there is a continuum of taxes and validity periods T that yield the same equilibrium. Differentiating the resulting system and computing the effects on household welfare in (8), an increase in taxes or a fall in T (both corresponding to an increase in $\hat{T}$), leading to an increase in g, results in a fall in welfare.[20,21]

The effects of changes in $\hat{T}$ on lord spending is less clear-cut due to Laffer curve effects. Using the resource constraint (20) and equilibrium price changes (29), the relationship between $\hat{T}$ and g is determined by the household optimality conditions (14)–(15) and is $u'(\xi - g)(1-\beta\hat{T}) = v'(\xi - g/(1-\hat{T}))$. The effect of a change in $\hat{T}$ on g is

$$\frac{dg}{d\hat{T}} = \frac{1}{\hat{T}^2} \frac{u'(\xi - g)\beta - v''\left(\xi - \frac{\hat{T}}{\hat{T}-1}g\right)\left(\frac{\hat{T}^2}{\left(\hat{T}-1\right)^2}g\right)}{\left(u''(\xi - g)\frac{\hat{T}-\beta}{\hat{T}} - \frac{\hat{T}}{\hat{T}-1}v'\left(\xi - \frac{\hat{T}}{\hat{T}-1}g\right)\right)}. \tag{31}$$

The sign cannot be determined, although for e.g., τ close to zero so that $\hat{T}$ is close to one, revenues are increasing since the second term in the numerator then dominates. When taxes are so high that households do not re-coin, i.e., $1-\tau < \bar{\chi}^T$, then, using that $p_t/p_{t-1} = 1/\bar{\chi}$ and expressions (14)–(16) and (29), revenues, and hence lord spending, depend only on confiscations of illegal coins, which is independent of $\hat{T}$.

5 Short-lived or long-lived currencies

This section analyzes a model with long-lived coins and compares it with the periodic re-coinage system described above. To generate revenues in the system with long-lived coins

[20] From (14), (20) and (29), letting $a = u''(\bar{c}_1 + \bar{c}_2)\left(1 - \beta\hat{T}\right)$ and $b = \hat{T}v''(\bar{c}_2)$, we have $\frac{d\bar{c}_1}{d\hat{T}} = -\frac{\bar{c}_1}{\hat{T}-1}$ and

$$\frac{d\bar{c}_2}{d\hat{T}} = \frac{u'(\bar{c}_1 + \bar{c}_2) - v'(\bar{c}_2)}{a-b} + \frac{a}{\left(\hat{T}-1\right)(a-b)}\bar{c}_1. \tag{30}$$

Using that (8) is $\frac{1}{1-\beta}\left(u(\bar{c}_1 + \bar{c}_2) - v(\bar{c}_2)\right)$ and differentiating establishes the result.

[21] There are potentially more than one $\hat{T}$ leading to the same spending level. However, for any $\hat{T}'$ and $\hat{T}''$ leading to the same spending level, household welfare is always highest at the lowest $\hat{T}$, since an increase in $\hat{T}$ always leads to an increase in $\bar{c}_2$, implying that $\bar{c}_2$ is lower at the lowest $\hat{T}$.

where all coins are legal tender, the lord debases the coins over time. Specifically, we adapt the model in Sussman and Zeira (2003) to the setting described above, where debasement is modelled so that the amount of silver in coins, denoted b_t, decreases according to $b_t = \frac{b_{t-1}}{1+\pi}$.[22] As above, household preferences are given by (8), and the household faces the CIA constraint $p_t c_{1t} = m_t$. Note that household money holdings now consist of coins minted in different periods with different silver content. Let $n_{t,r}$ denote coins surviving in period t that were minted in period r. Then, money holdings are $m_t = \sum_0^t n_{t,r}^n$. In period t, households hand in the amount $\mu_{t,r}$ of coins that were minted in period $r \leq t$. Clearly, $\mu_{t,r} \leq n_{t,r}$ and $n_{t+1,r} = n_{t,r} - \mu_{t,r}$ remain in period $t+1$. Given that a household hand in the amount $r_t^h = \sum_0^t \mu_{t,r}$, it receives $(1-\tau)\, n_t^h$ in new debased coins, where $n_t^h = \sum_0^t b_r \mu_{t,r}/b_t$. The budget constraint is then $m_{t+1} = \Pi_t + m_t - r_t^h + (1-\tau)\, n_t^h - p_t c_{1t} - p_t c_{2t}$. The household can test the silver content of coins costlessly once every period and hence will hand in the coins with the highest silver content for re-minting. Then, only the coins minted in the last T periods remain in circulation in period t. Letting s_t denote the (mint) price of silver, coins from period $r < t$ that satisfy $s_t b_r \geq 1$ are handed in for re-minting and coins from period $r' < t$ where $s_t b_{r'} < 1$ are kept by households. In equilibrium, where lord revenues are positive and hence minting and melting are positive as well, the mint price of silver is $s_t = (1-\tau)/b_t$. Then, the conditions for whether to re-mint or not can be summarized by a cutoff value T that satisfies

$$\frac{b_{t-T}}{b_t}(1-\tau) \geq 1 \text{ and } \frac{b_{t-T+1}}{b_t}(1-\tau) < 1. \tag{32}$$

The household first-order condition with respect to m_t is

$$\frac{p_t}{p_{t-1}} = \beta \frac{u'(c_t)}{u'(c_{t-1}) - v'(c_{2t-1})}. \tag{33}$$

Using that we have $r_t = \sum_0^t b_r \mu_{t,r}$ in equilibrium, government revenues are $\tau/b_t \sum_0^t b_r \mu_{t,r}$.

Due to debasement, melted coins from period $t-T$ generate $(1+\pi)^T$ coins in period t. Hence, the number of coins in cohort t is $n_{t,t} = (1+\pi)^T n_{t-T,t-T}$, and, since we restrict attention to steady states, we have $n_{t,t-u} = (1+\pi)^u \pi/(1+\pi-(1+\pi)^{1-T}) m_t$. Using this, m_t evolves according to $m_{t+1} = (1+\pi)\, m_t$. Government spending is, using the evolution

[22] For simplicity, we ignore exports and imports since these are zero in the periodic re-coinage case.

of b_t, the CIA constraint and $n_{t,t-T} = \mu_{t,t-T}$,

$$g_t = (1+\pi)^T \tau w c_{1t}, \tag{34}$$

where $w = \pi/((1+\pi)^T - 1)$ denotes the share of m_t that is re-minted. The equilibrium is then given by (33), (34) and the resource constraint $c_{1t} + c_{2t} + g_t = \xi_t$.

5.1 Optimal lord spending

In the debasement system, there is a fixed cost of upholding debasement, e.g., due to an increasing share of base metals in the coins. This fixed cost is denoted by C^d and paid for simplicity by the lord. Also, in the system with periodic re-coinage, due to monitoring in order to find illegal coins, there is a fixed monitoring cost, denoted C^p. Let $g^{\max}$ denote the maximum spending level, i.e., the highest g that satisfies (33), the resource constraint and (34). To model fiscal choices, we restrict attention to the case when the lord has a unique preferred spending level. Specifically, the payoff of the lord of consuming g is given by $z(g,\theta)$, where $\partial^2 z/\partial g^2 < 0$ and $\theta \in \Theta \subseteq R$ is a parameter affecting lord spending preferences. We restrict attention to the case when z has a maximum in $(0, g^{\max})$ to ensure an interior solution for g, i.e., $\frac{\partial z(0,\theta)}{\partial g} > 0$ and $\frac{\partial z(g^{\max},\theta)}{\partial g} < 0$ for all $\theta \in \Theta$. Also, we assume that $\partial^2 z/\partial g \partial \theta > 0$ so that the maximizer is increasing in θ.[23]

The lord chooses debasement π and τ in order to maximize

$$\sum_{t=0}^{\infty} \beta^t \left(z(g_t, \theta) - C^i \right), \tag{35}$$

where $i \in \{d, p\}$ subject to the relevant constraints, i.e., in the debasement case, the

[23] Alternatively, letting Z be a strictly concave function, we could assume that the lord cares about household welfare and that the objective is

$$Z(g_t, \theta) + \kappa \left[u(c_t) - v(c_{2t}) \right] - C^i,$$

where $\kappa > 0$. Using that we can in a cyclical equilibrium (or steady state in the debasement case) solve for $\bar{c}$ and $\bar{c}_2$ as functions of g from (14)–(15), (29) and the resource constraint, the first-order condition is

$$Z'(g,\theta) + \kappa \left[-u'(\xi - g) - v'(\bar{c}_2) \frac{d\bar{c}_2}{dg} \right].$$

Assuming that the second term is decreasing in g, which holds when $\frac{d^2 \bar{c}_2}{dg^2}$ is not too large, this establishes that the objective is strictly concave. If we define $z(g,\theta) = Z(g,\theta) + \kappa \left[u(\bar{c}) - v(\bar{c}_2) \right]$, then, under suitable conditions on Z, z satisfies the conditions in the main text.

resource constraint, (33) and (34). Note that there might be more than one tax rate yielding the same spending level in the model due to Laffer curve effects; see section 4.1. Since household welfare is decreasing in the effective tax rate $\hat{T}$, the analysis is restricted to the case when higher taxes lead to an increase in revenues, i.e., $\frac{dg}{d\hat{T}} > 0$ in (31).

We restrict attention to steady states in the debasement case.[24] In equilibrium, from the condition when old issues are re-minted in (32), we have $(1+\pi)^T(1-\tau) \geq 1$. Since, for a given level of spending, choosing π and τ so that this condition holds with equality increases household welfare, we restrict attention to such equilibria.[25] Then, using $(1+\pi)^T(1-\tau) = 1$, we can write (34) as $g = \pi\bar{c}_1$.

A key observation is that the equilibrium with debasement level π and tax τ is equivalent in terms of spending to an equilibrium under periodic re-coinage, when the condition for only using new coins in Theorem 1 holds. Specifically, let $g^{d*}(\theta)$ denote the optimal spending level under debasement, given θ, and let $\tau^{d*}(\theta)$ and $\pi^*(\theta)$ be the corresponding values of taxes and debasement. For any $\pi^*(\theta)$, choose the Gesell tax and period of legality, denoted by τ^p and T^p, so that $1+\pi^*(\theta) = (1-\tau^p)^{-\frac{1}{T^p}}$. Then, as long as $\bar{\chi} < \frac{1}{1+\pi}$, households hold only new coins in the periodic re-coinage case, and, using Theorem 1, the household money holding optimality condition (33) in the debasement case coincides with (14) and (15) in the periodic re-coinage case. Also, since $1+\pi^*(\theta) = (1-\tau^p)^{-\frac{1}{T^p}} = \hat{T}^p$, using (29) with $p_t/p_{t-1} = \hat{T}^p$ and $g = \pi\bar{c}_1$, periodic re-coinage and debasement yield the same spending levels. Hence, the private sector allocation is the same in the two systems. On the other hand, if $\bar{\chi} \geq \frac{1}{1+\pi}$, i.e., when households hold illegal coins in the system of periodic re-coinage, revenues are unaffected by τ and T, see section 4.1. Specifically, let $\hat{g}^{p*}$ denote the upper bound of lord revenues under periodic re-coinage, i.e., when τ and T satisfies $(1-\tau)^{\frac{1}{T}} = \bar{\chi}$ and $\hat{\theta}$ the corresponding lord preference parameter. Allocations

[24] And to cyclical equilibria in the re-coinage case.

[25] To see this, suppose $(1+\pi)^T(1-\tau) > 1$ and change π and τ so that g is constant. Consider τ, $1+\pi$ and $\bar{c}_2$ with $\bar{c} = \bar{c}_1 + \bar{c}_2 = \xi - g$ being constant. Expressions (33) and (34) can be written as

$$\begin{aligned} 1+\pi &= \beta\frac{u'(\bar{c})}{u'(\bar{c}) - v'(\bar{c}_2)} \\ \xi - \bar{c} &= (1+\pi)^T\tau\frac{1+\pi}{(1+\pi)^T - 1}(\bar{c} - \bar{c}_2). \end{aligned} \tag{36}$$

A reduction in $1+\pi$ and change in τ so that the second expression holds is feasible when $(1+\pi)^T(1-\tau) > 1$. Differentiating the first expression in (36) with respect to $1+\pi$ and $\bar{c}_2$ establishes that $\frac{d\bar{c}_2}{d(1+\pi)} > 0$. Since household steady-state payoff is $(u(\bar{c}) - v(\bar{c}_2))/(1-\beta)$, a decrease in $1+\pi$ and corresponding change in τ such that g and $\bar{c}$ are constant in the second expression in (36) increases household payoff.

are then different in the two systems as long as $g^{d*}(\theta) > \hat{g}^{p*}$. The decision whether to use periodic re-coinage or debasement depends partly on the fixed cost of operating the two systems and partly on whether the desired spending level is sufficiently high.

Theorem 3 *If $C^p > C^d$, all lords choose debasement, while if $C^p < C^d$, lord types $\theta \leq \hat{\theta}$ choose periodic re-coinage and types $\theta > \hat{\theta}$ where*

$$z\left(\hat{g}^{p*}, \theta\right) - C^p \leq z\left(g^{d*}\left(\theta\right), \theta\right) - C^d \tag{37}$$

weakly prefer debasement. Let $\bar{\theta}$ denote the value of θ where expression (37) holds with equality. We have $\bar{\theta} > \hat{\theta}$.

An implication when $C^d < C^p$ is that the set of lord types in $(\hat{\theta}, \bar{\theta})$ strictly prefers periodic re-coinage but chooses τ and T so that we in equilibrium have $(1-\tau)^{\frac{1}{T}} < \bar{\chi}$. To see this, note that the lord type where $g^{d*}(\hat{\theta}) = \hat{g}^{p*}$ strictly prefers periodic re-coinage. Since $\partial^2 z / \partial g \partial \theta > 0$, and, hence, optimal lord spending level is increasing in θ, all types in the interval $(\hat{\theta}, \bar{\theta})$ also strictly prefer periodic re-coinage. Thus, these lord types prefer periodic re-coinage despite the fact that households do not hand in coins for re-coinage and illegal coins circulate along with legal currency.

Another mechanism that potentially drives changes in monetary systems is changes in the cost of using the non-cash alternative. We model changes in the cost of using the non-cash alternative by letting $v\left(c_{2t}\right) = Kw\left(c_{2t}\right)$ and varying K. Denote by $\hat{c}_1$ the value of $\bar{c}_1$ of the lord type $\hat{\theta}$ at K. We have the following result.

Theorem 4 *If $C^d > C^p$, the set of lord preference parameters θ that results in an optimal choice of a system of periodic re-coinage becomes larger when the cost of the non-cash alternative increases.*

Intuitively, since a larger share of transactions is made in the market[26], in turn leading to higher revenues for the lord, the increase in the cost of the non-cash alternative makes periodic re-coinage more viable.

[26] For a given g, differentiating the equilibrium conditions, it is easy to show that an increase in K leads to an increase in $\bar{c}_1$ and a reduction in $\bar{c}_2$.

6 Relationship to empirical evidence

Due to the scarcity of data, it is difficult to match the model to the empirical evidence. However, the results in Theorems 1 and 2 can be judged relative to the evidence in section 2.3. The empirical evidence indicates that new coins almost exclusively circulated in England during a period when withdrawals occurred relatively infrequently (973–1035). After 1035, the intervals became shorter, tightening the cutoff in the theorem, and if the fee was unchanged, the shorter intervals also increased the implied yearly fee. Before 1035, 83 percent of the hoards contain only the last issue whereas only 33 percent after 1035; see Svensson (2016), Table 2. Regarding the number of coins from different issues in the hoards, the pattern is similar. Before 1035, the share of the last type is 86 percent, and, after 1035, the share drops to 54 percent. There is similar evidence from Thuringia in Germany, where the tax was 25 percent and withdrawals occurred every year: the coin hoards usually contain several types; see Svensson (2016), Table 3. The share of hoards that contains only the last type is 2.4 percent, whereas the vast majority of hoards–more than 80 percent–contains three types or more. Note that this can still be consistent with optimal lord behavior since higher operating costs of debasements can induce lords to operate periodic re-coinages where illegal coins circulate; see section 5.1. Regarding prices, the evidence is scarce. However, evidence of price regulation from the Frankish empire in the late 8$^{\text{th}}$ century seems to indicate that prices rose during a cycle.

Empirical observations show that periodic re-coinage broke down in England in the beginning of the 12$^{\text{th}}$ century and in Germany in the end of the 13$^{\text{th}}$ century, and long-lived coins were introduced. In light of section 5, increases in fiscal spending (due to an increase in θ) tend to induce a switch to a system with long-lived coins. An alternative explanation is the increase in the cost of the non-cash alternative to coins since bartering became more costly when the complexity of economies increased. However, as argued in section 5, this tends to make periodic re-coinage more rather than less attractive.

7 Discussion

Several simplifying assumptions have been used when modelling periodic re-coinage. First, we rule out wear, clipping and sweating of coins. Wear and tear of coins, clipping and sweating implies that coins handed in and re-minted did not need to have full intrinsic value, and the actual Gesell tax might therefore have been effectively lower than the official level. Second, there are no incentives to hoard coins for e.g., precautionary motives. In a previous version of the model, agents could transform silver into jewelry, see Svensson and Westermark (2016). Jewelry can also be sold at the market to relax the CIA constraint. But, money also has a liquidity value to households, so the CIA constraint will still bite. Jewelry is then a store of value for the households with properties similar to hoarding since jewelry gives the households a benefit, as would precautionary savings due to hoarding. In this setup, the results are very similar to the results above: the cutoff when using only new coins is the same and prices evolve in a similar fashion. Third, the only source of revenue is the Gesell tax. In practice, other sources were available. A way of extending the model in this direction would be to add a distortionary tax on the endowment (τ^{ξ}), where we model the distortion so that a part of the tax revenue is wasted. This would not matter (qualitatively) in sections 3–4 since results in these sections are derived for a given τ and T, but the cutoff condition in Theorem 1 has to be modified to take τ^{ξ} into account.[27] Finally, we abstract from economic growth and exposure to international trade, besides in silver, that could affect the choice of monetary system. In particular, the possibility to use international coins could make it more difficult to sustain periodic re-coinage.

Also worth noting is that Gesell taxes used in the Middle Ages had a different purpose than in the current discussion on Gesell taxes, see e.g., Buiter and Panigirtzoglou (2003). The current debate focuses on how to alleviate the lower bound on interest rates, while the system in use during the Middle Ages had a fiscal purpose.

8 Conclusions

A frequent method for generating revenue from seigniorage in the Middle Ages was to use Gesell taxes through periodic re-coinage, where coins are legal only for a limited period of time. In such a short-lived coinage system, old coins are declared invalid and exchanged for new coins at publicly announced dates and exchange fees, similar to Gesell taxes. Empirical evidence shows that re-coinage could occur as often as twice per year

[27] In the modified model, the corresponding results would be made for a given τ, T and τ^{ξ}; the household endowment after tax is $\left(1-\tau^{\xi}\right)\xi$, and the government gets revenues $\tau^{\xi}\xi - k\left(\tau^{\xi}\right)$ from the tax where $k\left(\tau^{\xi}\right)$ are the "wasted" revenues.

in a currency area during the Middle Ages. Although the short-lived coinage system was predominant for almost 200 years in large parts of medieval Europe, it has seldom, if ever, been mentioned or analyzed in the literature of economics.

The main purpose of this study is to discuss the evidence for and analyze the consequences of short-lived coinage systems. A cash-in-advance model is formulated to capture the implications of this monetary institution. The model includes households, firms and a lord, where households care about cash and credit goods. Households can hold both new and old coins, and the choice of which coins to hold is endogenous. The lord receives seigniorage from re-coinage fees, which are used to finance lord consumption.

The system with Gesell taxes works 1) if the tax is sufficiently low, 2) if the period of time between two instances of re-coinage is sufficiently long, and 3) if the probability of being penalized for using old illegal coins is sufficiently high. Prices increase during an issue period and fall immediately after the re-coinage date. Moreover, the higher the Gesell tax is, the higher the price increases (as long as the coins are surrendered for re-coinage). Periodic re-coinage ceased to be used after 150-200 years. In the model, increased fiscal spending tends to induce the lord to switch to systems with long-lived coins since these systems can generate higher revenues. On the other hand, an increase in the cost of the non-cash alternative, e.g., bartering, tends to make periodic re-coinage more viable, since more transactions are made in the market, in turn leading to higher lord revenues.

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A Appendix

A.1 Proofs

Proof of Lemma 1:

Note that, when analyzing e.g. money holdings in a cycle, the period where the fee is levied is important. Thus, when comparing a time period t to a point in the cycle, the notation mod (t) should be used, with mod $(t) \in \{1, \ldots, T\}$. However, instead of writing e.g. mod $(t) < T$, we often write $t < T$ and so on.

Subcase 1. $\mathrm{Im}_t < 0$ and $\mathrm{Im}_{t+s} > 0$.

Let $\bar{c} = \xi - g$ and note that $c_r = \xi - g + \mathrm{Im}_r$. Since $\mathrm{Im}_t < 0$ ($\mathrm{Im}_{t+s} > 0$) implies $n_t^n > 0$ ($\mu_{t+s}^n > 0$), we have $p_t = 1 - \tau$ and $p_{t+s} = 1$ and $c_t < c_{t+s}$. Without loss of generality, suppose t $(t + s)$ is the smallest (largest) time period when exports (imports) are negative (positive), i.e., $\mu_r^n = n_r^n = 0$ for $r < t$ and $r > t + s$.

Consider prices p_r, p_{r+1} such that $r \geq t + s$. Note in particular that we have $p_{t+s+1} \leq p_{t+s}$. The CIA constraints when imports are zero are

$$\begin{aligned} p_{t+r} \left(c_{1t+r} + g - \mathrm{Im}_{t+r}\right) &= p_{t+r} \left(\xi - c_{2t+r}\right) = m_{t+r+1}^n \qquad \text{(A.1)} \\ p_{t+r+1} \left(c_{1t+r+1} + g - \mathrm{Im}_{t+r+1}\right) &= p_{t+r+1} \left(\xi - c_{2t+r+1}\right) = m_{t+r+2}^n = m_{t+r+1}^n + p_{t+r+1} g \end{aligned}$$

a) Suppose $t + s < T$. Consider $r = t + s + 1, \ldots, \mathrm{mod}\,(t-1)$. Then, for any r, $\mathrm{Im}_r = 0$ and hence $c_r = \bar{c}$. In general, if $p_r \leq p_{r-1}$ (and, when $r = T$, $p_{T+1} \leq (1 - \tau) p_T$) then, from the CIA constraint (A.1), $c_{2r} < c_{2r-1}$ implying that $v'(c_{2r-1}) > v'(c_{2r})$. Hence, setting $r - 1 = t + s$ and using that imports are zero for periods r and $r + 1$ so that $c_{t+s} \geq c_r = c_{r+1}$, we have

$$\frac{p_{r+1}}{p_r} = \frac{\beta u'(c_{r+1})}{u'(c_r) - v'(c_{2r})} < \frac{\beta u'(c_r)}{u'(c_{r-1}) - v'(c_{2r-1})} = \frac{p_r}{p_{r-1}} \qquad \text{(A.2)}$$

when $r \neq T$ and

$$\frac{1}{1-\tau} \frac{p_{r+1}}{p_r} = \frac{\beta u'(c_{r+1})}{u'(c_r) - v'(c_{2r})} < \frac{\beta u'(c_r)}{u'(c_{r-1}) - v'(c_{2r-1})} = \frac{p_r}{p_{r-1}} \qquad \text{(A.3)}$$

when $r = T$ and $\mathrm{Im}_{r+1} = 0$.

If $t \geq 2$ then, by induction $p_1 < 1 - \tau$, a contradiction.

If $t = 1$ then $n_1^n > 0$ so that $\mathrm{Im}_1 < 0$ and hence $p_1 = 1 - \tau$. Note that, using $p_{t+s} = 1$,

$$p_{t+s} = (\xi - c_{2t+s}) = m_{t+s+1}^n \iff c_{2t+s} = \xi - \frac{1}{p_{t+s}} m_{t+s+1}^n = \xi - m_{t+s+1}^n \qquad \text{(A.4)}$$

and using (A.1) and (18) and $p_1 = 1-\tau$,

$$\begin{aligned} c_{21} &= \xi - \frac{1}{p_1}\left(m_1^n + p_1 g + (1-\tau)\, n_1^n\right) && \text{(A.5)} \\ &= \xi - g - \frac{1}{p_1}(1-\tau)\, n_1^n - \frac{1}{p_1}(1-\tau)\left(m_{t+s+1}^n + \sum_{r=t+s+1}^{T} p_r g\right) < c_{2t+s} \end{aligned}$$

Consider $\hat{t}$ such that $\mathrm{Im}_{\hat{t}} \geq 0$ and $\mathrm{Im}_r < 0$ for $r = 1, \ldots, \hat{t}-1$. Then, if $\hat{t} > 2$, $p_{r-1} = p_r = 1-\tau$ and, from the CIA constraint (9),

$$\begin{aligned} p_{r-1}\left(\xi - c_{2r-1}\right) &= m_r^n && \text{(A.6)} \\ p_r\left(\xi - c_{2r}\right) &= m_{r+1}^n = m_r^n + p_r g + (1-\tau)\, n_r^n, \end{aligned}$$

we get $c_{2r} < c_{2r-1}$. Then $c_{2r} < c_{2t+s}$ and hence $c_{2t+s} > c_{2\hat{t}-1}$. If $\hat{t} = 2$ then, from (A.5), we have $c_{2t+s} > c_{2\hat{t}-1}$. Since $\mathrm{Im}_{\hat{t}-1} < 0$ and $\mathrm{Im}_{\hat{t}} \geq 0$, $c_{\hat{t}-1} < c_{t+s}$ and $c_{t+s+1} \leq c_{\hat{t}}$. Using $c_{21} < c_{2t+s}$, $c_{2t+s} > c_{2\hat{t}-1}$, $c_{\hat{t}-1} < c_{t+s}$ and $c_{t+s+1} \leq c_{\hat{t}}$ it follows that

$$\frac{p_{t+s+1}}{p_{t+s}} = \frac{\beta u'\left(c_{t+s+1}\right)}{u'\left(c_{t+s}\right) - v'\left(c_{2t+s}\right)} > \frac{\beta u'\left(c_{\hat{t}}\right)}{u'\left(c_{\hat{t}-1}\right) - v'\left(c_{2\hat{t}-1}\right)} = \frac{p_{\hat{t}}}{p_{\hat{t}-1}} \tag{A.7}$$

a contradiction, since $p_{t+s+1} \leq p_{t+s}$ and $p_{\hat{t}} \geq p_{\hat{t}-1}$.

b) Suppose $t+s = T$ so that $p_T = 1$. Let $\hat{t}$ be the time period where $n_r^n > 0$ for $r = t, \ldots, \hat{t}-1$ and $n_{\hat{t}}^n = 0$. Note that

$$\frac{\beta u'\left(c_t\right)}{u'\left(c_{t-1}\right) - v'\left(c_{2t-1}\right)} = \frac{p_t}{p_{t-1}} \leq 1 \tag{A.8}$$

when $t > 1$ and, since $c_t < c_{\hat{t}}$, $c_{t-1} > c_{\hat{t}-1}$ and, using a similar argument as in (A.6), $c_{2\hat{t}-1} < c_{2t-1}$ we get

$$\frac{p_{\hat{t}}}{p_{\hat{t}-1}} = \frac{\beta u'\left(c_{\hat{t}}\right)}{u'\left(c_{\hat{t}-1}\right) - v'\left(c_{2\hat{t}-1}\right)} < \frac{\beta u'\left(c_t\right)}{u'\left(c_{t-1}\right) - v'\left(c_{2t-1}\right)} \leq 1 \tag{A.9}$$

contradicting $p_{\hat{t}} \geq p_{\hat{t}-1}$. When $t = 1$ we get, since $p_T = 1$ and $p_t = p_1 = 1-\tau$ and

$$\frac{\beta u'\left(c_1\right)}{u'\left(c_T\right) - v'\left(c_{2T}\right)} = \frac{1}{1-\tau}\frac{p_1}{p_T} = 1 \tag{A.10}$$

Then, proceeding along the lines of (A.5) establishes that $c_{2\hat{t}-1} < c_{2T}$. Using that $c_{\hat{t}-1} < c_T$

and $c_{T+1} < c_{\hat{t}}$, we have

$$\frac{p_{\hat{t}}}{p_{\hat{t}-1}} = \frac{\beta u'\left(c_{\hat{t}}\right)}{u'\left(c_{\hat{t}-1}\right) - v'\left(c_{2\hat{t}-1}\right)} < \frac{\beta u'\left(c_{T+1}\right)}{u'\left(c_T\right) - v'\left(c_{2T}\right)}, \tag{A.11}$$

and we can again establish a contradiction.

Subcase 2. $\mathrm{Im}_t > 0$ and $\mathrm{Im}_{t+s} < 0$.

Since $\mathrm{Im}_t > 0$ ($\mathrm{Im}_{t+s} < 0$) implies $\mu_t^n > 0$ ($n_{t+s}^n > 0$), we have $c_t > c_{t+s}$, $p_t = 1$ and $p_{t+s} = 1 - \tau$. Choose t and $t+s$ so that $\mathrm{Im}_r = 0$ for $r = t+1, \ldots, t+s-1$, implying $n_r^n = \mu_r^n = 0$.[28] Also, for any r, $\mathrm{Im}_r = 0$ and hence $c_r = \bar{c}$.

Suppose $t + s < T$. In general, using the CIA constraints (A.1) as in Subcase 1, if $p_{r+1} \leq p_r$ (and, when $r = T$, $p_{T+1} \leq (1-\tau)\, p_T$) then, from the CIA constraint $c_{2r+1} < c_{2r}$ implying that $v'\left(c_{2r}\right) > v'\left(c_{2r+1}\right)$.

Suppose there is some $\hat{t}$ such that $t < \hat{t} \leq T$ where $\mathrm{Im}_{\hat{t}} = 0$ so that $c_{\hat{t}} = \bar{c}$ and where $p_{\hat{t}} > p_{\hat{t}-1}$. Let $\hat{t}$ be the lowest such t. Then for any $r = t, \ldots, \hat{t}-1$, we have $p_r \leq p_{r-1}$ and hence, using (A.1) and (A.6) with

$$p_r\left(\xi - c_{2r}\right) = m_r^n + (1-\tau)\, n_r^n + p_r g, \tag{A.12}$$

we have $c_{2r} < c_{2r-1}$. By induction, using (A.1) when $n_r^n = 0$, $c_{2t} > c_{2\hat{t}-1}$. Note also that, from the choice of $\hat{t}$, $\mathrm{Im}_{\hat{t}-1} \leq 0$. Then, since $c_{\hat{t}} \geq c_{t+1}$, $c_{\hat{t}-1} \leq c_t$, we have

$$\frac{p_{\hat{t}}}{p_{\hat{t}-1}} = \frac{\beta u'\left(c_{\hat{t}}\right)}{u'\left(c_{\hat{t}-1}\right) - v'\left(c_{2\hat{t}-1}\right)} < \frac{\beta u'\left(c_{t+1}\right)}{u'\left(c_t\right) - v'\left(c_{2t}\right)} \leq 1, \tag{A.13}$$

a contradiction. Hence $p_T = 1 - \tau$.

Using a modified version of (A.5), we have $c_{2T} < c_{2t}$. Since, from the choice of t, $c_T < c_t$ and $c_1 \geq c_{t+1}$ we have

$$\frac{1}{1-\tau}\frac{p_1}{p_T} = \frac{\beta u'\left(c_1\right)}{u'\left(c_T\right) - v'\left(c_{2T}\right)} \leq \frac{\beta u'\left(c_{t+1}\right)}{u'\left(c_t\right) - v'\left(c_{2t}\right)} \leq 1 \tag{A.14}$$

implying that $p_1 = (1-\tau)\, p_T \leq (1-\tau)^2$, contradicting $p_1 \geq 1 - \tau$.

[28] If $n_{t+r}^n > 0$ then $\mu_{t+r}^n > 0$ for $\mathrm{Im}_{t+r} = 0$. Since $n_{t+r}^n > 0$ implies $p_{t+r} = 1 - \tau$ and $\mu_{t+r}^n > 0$ implies $p_{t+r} = 1$ we have a contradiction.

Suppose $t+s=T$. Then $p_T=1-\tau$ and we get

$$\frac{p_T}{p_{T-1}}=\frac{\beta u'(c_T)}{u'(c_{T-1})-v'(c_{2T-1})}\leq 1 \tag{A.15}$$

and, since $\mathrm{Im}_T<0$, we have $c_T<c_{T-1}$ and, proceeding as in (A.6), $c_{2T-1}>c_{2T}$. Then, using that $c_1\geq c_T$,

$$\frac{1}{1-\tau}\frac{p_1}{p_T}=\frac{\beta u'(c_1)}{u'(c_T)-v'(c_{2T})}<\frac{\beta u'(c_T)}{u'(c_{T-1})-v'(c_{2T-1})}\leq 1 \tag{A.16}$$

implying that, $p_1=(1-\tau)\,p_T\leq(1-\tau)^2$, a contradiction.■

Proof of Lemma 2:

Case 1**.** First, suppose that $t\neq T$. We have

$$p_t c_{1t}=m_t^n+e_t\chi(c_{1t})\,m_t^o. \tag{A.17}$$

Suppose that $\mu_t^o=0$. If $n_t^n>0$ then $p_t=1-\tau$ from (4) and thus, $\mu_t^n=0$ and $\mathrm{Im}_t=-n_t^n$. Using (3) and (18), we get

$$p_t\,(c_{1t}+g-\mathrm{Im}_t)=m_{t+1}^n+e_t\chi(c_{1t})\,m_t^o. \tag{A.18}$$

Suppose $n_t^n=\mu_t^n=0$. Since $n_t^n=\mu_t^n=0$ implies $\mathrm{Im}_t=0$, a similar argument holds in this case. Suppose that $\mu_t^n>0$ so that $p_t=1$ from (5). Using (3) and money transition (18) we get $p_t\,(c_{1t}+g-\mathrm{Im}_t)=m_{t+1}^n+e_t\chi m_t^o$.

A similar argument holds if $\mu_t^o>0$. We get, using $p_t=e_t$, $\mathrm{Im}_t=\mu_t^n+\mu_t^o-n_t^n$ and $m_{t+1}^o=\chi(c_{1t})\,m_t^o-\mu_t^o$,

$$p_t\,(c_{1t}+g-\mathrm{Im}_t)=m_{t+1}^n+e_t m_{t+1}^o. \tag{A.19}$$

Using the resource constraint establishes the result.

Case 2**.** Now, suppose that $t=T$.

Suppose that $\mu_t^o=0$. Suppose $n_t^n>0$**.** We have $m_{t+1}^n=(1-\tau)\,r_t^h$ and

$$r_t^h\in[0,m_t^n+p_t g+(1-\tau)\,n_t^n-\mu_t^n] \tag{A.20}$$

If r_t^h is equal to the upper bound, we can proceed as above to establish $p_t\left(c_{1t}+g-\mathrm{Im}_t\right)=\frac{1}{1-\tau}m_{t+1}^n$. If $r_t^h<1$ then $e_T\geq 1$ from (13) and thus, using (3), (18) and (19), we have, using the constraints imposed on p_t and e_t when minting or melting is positive gives

$$\begin{aligned} p_t\left(c_{1t}+g-\mathrm{Im}_t\right) &= \frac{1}{1-\tau}e_t m_{t+1}^n+\left(1-e_t\right)\left(m_t^n+p_t g\right) \\ &\quad -\left(e_t-1\right)\left(1-\tau\right)n_t^n+\left(e_t-1\right)\mu_t^n+e_t m_{t+1}^o \end{aligned} \tag{A.21}$$

A similar argument holds if $\mu_t^n>0$, if $\mu_t^o>0$ and if $\mu_t^n=\mu_t^o=n_t^n=0$. If r_t^h is interior then, from (13), $e_T=1$ implying that $p_t\left(c_{1t}+g-\mathrm{Im}_t\right)=\frac{1}{1-\tau}m_{t+1}^n+e_t m_{t+1}^o$. Using the resource constraint establishes the result. ■

Proof of Corollary 1:

Fix g at it's equilibrium value. We have, using the CIA constraint (9), money transition (18) and (20), when $t\neq 1$,

$$p_t=\frac{\xi-c_{2t-1}}{\xi-c_{2t}-g}p_{t-1} \tag{A.22}$$

and, when $t=1$,

$$p_1=\frac{\xi-c_{2T}}{\xi-c_{21}-g}\left(1-\tau\right)p_T \tag{A.23}$$

Using (14) and (15) gives

$$\frac{\beta u'\left(\bar{c}\right)}{u'\left(\bar{c}\right)-v'\left(c_{2t-1}\right)}=\frac{\xi-c_{2t-1}}{\xi-c_{2t}-g}. \tag{A.24}$$

Then

$$v'\left(c_{2t-1}\right)=u'\left(\bar{c}\right)\left(1-\beta\frac{\xi-c_{2t}-g}{\xi-c_{2t-1}}\right). \tag{A.25}$$

Suppose that there are t and r such that $c_{2t}>c_{2r}$. Then there is some s such that $c_{2s}>c_{2s+1}$ and $c_{2s+1}\leq c_{2s+2}$. Hence,

$$\frac{\xi-c_{2s+2}-g}{\xi-c_{2s+1}}<\frac{\xi-c_{2s+1}-g}{\xi-c_{2s}} \tag{A.26}$$

From (A.25), this contradicts $v'\left(c_{2s}\right)>v'\left(c_{2s+1}\right)$. Hence, $c_{2t}=c_{2r}$ for all t,r.■

Proof of Theorem 1:

Lemma 1 implies that $n_t^n=\mu_t^n=0$. From Corollary 1, $c_{1t}=\bar{c}_1$ for all t and hence we define $\bar{\chi}=\chi\left(\bar{c}_1\right)$.

Step 1. Since $r_T^h=m_T^n+p_T g$ we have, from (7), (12) and the household optimality condition for r^h_T, that $q_T = \frac{1}{1-\tau}$, $e_1\bar{\chi} \le \frac{e_T}{q_T}$, $e_{t+1}\bar{\chi} \le e_t$ and $e_T = 1$ and hence

$$\frac{e_T}{q_T} \ge e_1\bar{\chi} \ge e_2\bar{\chi}^2 \ge \ldots \ge e_T\bar{\chi}^T \iff 1-\tau \ge \bar{\chi}^T. \tag{A.27}$$

Step 2. Prices.

We have, using (9) and (18), for $t \neq T+1$,

$$\frac{\bar{c}_1}{\bar{c}_1+g} m^n_{t+1} = m^n_t \tag{A.28}$$

and, using (9) and (17),

$$\tau r^h_T = \sum_{t=1}^{T} p_t g = \sum_{t=1}^{T} m^n_t \frac{g}{\bar{c}_1} = \frac{g}{\bar{c}_1} \sum_{t=1}^{T} \left(\frac{\bar{c}_1}{\bar{c}_1+g}\right)^{T-t} m^n_T \tag{A.29}$$

so that, using $p_T = \frac{m^n_T}{\bar{c}_1}$ and that $r^h_T = m^n_T + p_T g = m^n_T \frac{\bar{c}_1+g}{\bar{c}_1}$, the above expression is

$$\tau \frac{\bar{c}_1+g}{\bar{c}_1} = \frac{g}{\bar{c}_1} \sum_{t=1}^{T} \left(\frac{\bar{c}_1}{\bar{c}_1+g}\right)^{T-t} = \frac{\bar{c}_1+g}{\bar{c}_1} \left(1 - \left(\frac{\bar{c}_1}{\bar{c}_1+g}\right)^{T}\right) \tag{A.30}$$

and hence $\frac{\bar{c}_1}{\bar{c}_1+g} = (1-\tau)^{\frac{1}{T}}$ so that

$$\bar{c}_1 = (1-\tau)^{\frac{1}{T}} \left(\bar{c}_1 + g\right). \tag{A.31}$$

From (29), for $t = 2, \ldots, T$, we have

$$(1-\tau)^{\frac{1}{T}} p_t = p_{t-1} \tag{A.32}$$

and thus $p_1 = (1-\tau)^{\frac{T-1}{T}} p_T$.

Step 3. Computing $\bar{c}_1$, $\bar{c}_2$ and g.

From (14) we have

$$(1-\tau)^{-\frac{1}{T}} = \frac{\beta u'\left(\bar{c}_1+\bar{c}_2\right)}{u'\left(\bar{c}_1+\bar{c}_2\right) - v'\left(\bar{c}_2\right)} \tag{A.33}$$

and, from the resource constraint (20), we have

$$\bar{c}_1 + \bar{c}_2 + g = \xi. \tag{A.34}$$

Then equations (A.31), (A.33) and (A.34) determine $\bar{c}_1$, $\bar{c}_2$ and g. Since $p_T \leq 1$ from the optimality condition for melting new coins, any $p_1 \in [1-\tau, (1-\tau)^{\frac{T-1}{T}}]$ is possible, implying that $p_T \in [(1-\tau)^{\frac{1}{T}}, 1]$.

Step 4. Finding m_1^n.

Using the solution for $\bar{c}_1$ from step 2 and 3, m_1^n solves

$$p_T \bar{c}_1 = \frac{1}{1-\tau} m_1^n. \tag{A.35}$$

Then, for each $p_T \in [(1-\tau)^{\frac{1}{T}}, 1]$, there is a unique m_1^n that satisfies the CIA constraint.■

Proof of Theorem 2:

Preliminaries. From money transition (18), we have, except when $t = T$, using Lemma 2,

$$m_{t+1}^n = m_t^n \frac{\bar{c}_1 + g}{\bar{c}_1} + e_t \bar{\chi} m_t^o \frac{g}{\bar{c}_1}. \tag{A.36}$$

By assumption, c_{1t} is constant over the cycle and imports are zero.

Step 1. Exchange rates.

Using that $\mu_t^o = 0$ and, since $\mu_t^n = 0$ implies $m_t^n > 0$ for $t \neq 1$, that $e_t \bar{\chi} = e_{t-1}$ from (11) and (12) and, from the household optimality condition for r_T^h, $e_T \geq 1$, we have $e_t \geq \bar{\chi}^{T-t}$ and, using (11), $q_T e_{T+1} \bar{\chi} \geq e_T$. Moreover, if $r_T^h \in (0,1)$ then $e_T = 1$ implying that $e_t = \bar{\chi}^{T-t}$. Also, $q_T e_{T+1} \bar{\chi} = e_T$. Using (7) and combining establishes that $\bar{\chi}^T = 1 - \tau$ whenever $r_T^h \in (0,1)$. If $r_T^h = 0$ then $\bar{\chi}^T \geq 1 - \tau$.

Step 2. Showing $\bar{\chi} \leq \frac{\bar{c}_1}{\bar{c}_1 + g}$.

Since $\mu_t^o = 0$ for all t, we have $m_t^o = \bar{\chi} m_{t-1}^o$ for $t \neq 1$. Then, using (19) we have $m_1^o = \chi m_T^o + \left(m_T^n + p_T g - r_T^h\right)$ and $m_t^o = \bar{\chi} m_{t-1}^o$ and hence $m_1^o = \frac{1}{1-\chi^T}\left(m_T^n + p_T g - r_T^h\right)$ and, by repeatedly using $m_t^o = \bar{\chi} m_{t-1}^o$,

$$m_{t+1}^o = \frac{\bar{\chi}^t}{1-\bar{\chi}^T}\left(m_T^n + p_T g - r_T^h\right). \tag{A.37}$$

Government revenues during a cycle are, in terms of new coins, using (A.37),

$$\tau r_T^h + (1-\bar{\chi}) \sum_{t=1}^{T} m_t^o = \tau r_T^h + m_T^n + p_T g - r_T^h. \tag{A.38}$$

Now consider government expenditures. Using Lemma 2, that $m_t^n > 0$ for $t \neq 1$ since

$\mu_t^n = 0$ and new coin dividends are positive, that $e_{t-1} = \bar{\chi} e_t$ and that $m_t^o = \bar{\chi} m_{t-1}^o$ from (11), (12) and (19), we can write $p_t \bar{c}_1 = m_t^n + e_1 \bar{\chi} m_1^o$, we have

$$\sum_{t=1}^{T} p_t g = \frac{g}{\bar{c}_1 + g} \frac{\bar{c}_1 + g}{\bar{c}_1} \left(\sum_{t=1}^{T} m_t^n + T e_1 \bar{\chi} m_1^o \right). \tag{A.39}$$

Using that (A.36) gives $m_t^n = \frac{\bar{c}_1}{\bar{c}_1+g} m_{t+1}^n - \frac{g}{\bar{c}_1+g} e_1 \bar{\chi} m_1^o$, that $e_{t-1} = \bar{\chi} e_t$ and $m_t^o = \bar{\chi} m_{t-1}^o$ from (11) - (12) and repeatedly substituting gives

$$m_t^n = \left(\frac{\bar{c}_1}{\bar{c}_1 + g} \right)^{T-t} m_T^n - e_1 \bar{\chi} m_1^o \left(1 - \left(\frac{\bar{c}_1}{\bar{c}_1 + g} \right)^{T-t} \right). \tag{A.40}$$

Then, summing and equating expenditures with revenues, using (A.38) and (A.39) and we have $e_1 = \bar{\chi}^{T-1} e_T$ we get

$$\tau r_T^h + m_T^n + p_T g - r_T^h = \left(1 - \left(\frac{\bar{c}_1}{\bar{c}_1 + g} \right)^T \right) \left(m_T^n + p_T g + e_T \frac{\bar{\chi}^T}{1 - \bar{\chi}^T} \left(m_T^n + p_T g - r_T^h \right) \right). \tag{A.41}$$

This implies, using that, when $r_T^h > 0$ we have $e_T = 1$ and $1 - \tau = \bar{\chi}^T$, the following expression holds

$$\left(1 - \left(\frac{\bar{c}_1}{\bar{c}_1 + g} \right)^T \right) \left(\frac{1 - \bar{\chi}^T (1 - e_T)}{1 - \bar{\chi}^T} \right) = 1 \tag{A.42}$$

Suppose that $r_T^h > 0$. Then, from (13), $e_T = 1$ so that $\bar{\chi} = \frac{\bar{c}_1}{\bar{c}_1+g}$ and thus

$$\bar{c}_1 = (\bar{c}_1 + g) \bar{\chi}. \tag{A.43}$$

Suppose $r_T^h = 0$ so that $e_T \geq 1$. Letting $\tau^* = \frac{1-\bar{\chi}^T}{1-\bar{\chi}^T(1-e_T)}$ we have $\frac{\bar{c}_1}{\bar{c}_1+g} = (1 - \tau^*)^{\frac{1}{T}}$ and we can proceed as in Case 1 and thus

$$\bar{c}_1 = (\bar{c}_1 + g)(1 - \tau^*)^{\frac{1}{T}}. \tag{A.44}$$

When r_T^h is interior so that $\tau^* = 1 - \bar{\chi}^T$ prices evolve according to, for $t = 2, \ldots, T$,

$$\bar{\chi} p_t = p_{t-1} \tag{A.45}$$

and when $r_T^h = 0$, for $t = 2, \ldots, T$,

$$(1-\tau^*)^{\frac{1}{T}} p_t = p_{t-1}. \tag{A.46}$$

Note that, since $\tau^* \leq 1 - \bar{\chi}^T$ we have $\frac{\bar{c}_1}{\bar{c}_1+g} \geq \bar{\chi}$.

Step 3. Computing $\bar{c}_1$, $\bar{c}_2$ and g.

From (14) and (A.45),

$$\frac{1}{\bar{\chi}} = \frac{\beta u'(\bar{c}_1 + \bar{c}_2)}{u'(\bar{c}_1 + \bar{c}_2) - v'(\bar{c}_2)} \tag{A.47}$$

or, from (14) and (A.46),

$$(1-\tau^*)^{-\frac{1}{T}} = \frac{\beta u'(\bar{c}_1 + \bar{c}_2)}{u'(\bar{c}_1 + \bar{c}_2) - v'(\bar{c}_2)}. \tag{A.48}$$

Thus $\bar{c}_1$, $\bar{c}_2$ and g are determined by either (A.43), (A.47) and (A.34) or (A.44), (A.48) and (A.34). Since, using the optimality condition for melting new coins, $p_T \leq 1$ any $p_1 \in [1-\tau, \left(\frac{\bar{c}_1}{\bar{c}_1+g}\right)^T]$ is possible.

Step 4. Finding m_1^n.

Fix r_T^h and e_T. Using (9), (A.37) and the solution for $\bar{c}_1$ from step 2 and 3, m_T^n solves

$$m_T^n = p_T \frac{\bar{c}_1 - \frac{e_T \bar{\chi}^T}{1-\bar{\chi}^T} g}{1 + \frac{e_T \bar{\chi}^T}{1-\bar{\chi}^T}} + \frac{\frac{e_T \bar{\chi}^T}{1-\chi^T}}{1 + \frac{e_T \bar{\chi}^T}{1-\bar{\chi}^T}} r_T^h. \tag{A.49}$$

Then, for each $p_T \in [(1-\tau)\left(\frac{\bar{c}_1}{\bar{c}_1+g}\right)^{-T}, 1]$, there is a unique m_T^n that satisfies the CIA constraint.■

Proof of Theorem 3. The case when $C^p > C^d$ follows since the set of feasible spending levels is larger under debasement than under re-coinage. Suppose $C^p < C^d$ and let $\bar{\theta}$ denote the value of θ where expression (37) holds with equality. Note that, if $\theta \leq \hat{\theta}$ optimal spending choices under debasement and periodic re-coinage coincide, and since $C^p < C^d$ we have $\bar{\theta} > \hat{\theta}$. Importantly, since $\partial^2 z / \partial g \partial \theta > 0$ and hence $\frac{\partial z(\hat{g}^{p*},\theta)}{\partial \theta} < \frac{\partial z\left(g^{d*}(\theta),\theta\right)}{\partial \theta}$ for $\theta > \hat{\theta}$, an increase in fiscal preferences of the lord at $\bar{\theta}$ so that the desired spending level increases induces a switch from periodic re-coinage to debasement. This also implies that $\bar{\theta}$ is unique. Also, since $C^p < C^d$ and z is increasing in g for $\theta \leq \bar{\theta}$, we have $g^{d*}(\bar{\theta}) > \hat{g}^{p*}$.

■

Proof of Theorem 4. Consider the lord type that chooses spending so that $\bar{c}_1$ is unchanged at $\hat{c}_1$ when K increases under periodic re-coinage, i.e., the choice of g satisfies (14)-(15), (27) and the resource constraint and leads to the same household consumption of the cash good. Potentially, this might violate the cutoff condition for holding only new coins. To see that this is not the case, let the spending at the cutoff $\bar{\theta}$ for a given K be denoted as $\hat{g}^{p*}(K)$ and set g and $\hat{T}$ so that $\bar{c}_1$ is unchanged at $\bar{c}_1 = \hat{c}_1$. Differentiating (14)-(15), (27) and the resource constraint, treating $\hat{c}_1$ as fixed and letting $a = \left(\hat{T} - \beta\right) u''(\hat{c}_1 + \bar{c}_2) - \hat{T}Kw''(\bar{c}_2) < 0$ and $b = u'(\bar{c}_1 + \bar{c}_2) - Kw'(\bar{c}_2) > 0$, gives

$$\begin{aligned} \frac{dg}{dK} &= -\frac{\hat{c}_1}{b - a\hat{c}_1}\hat{T}w'(\bar{c}_2) \\ \frac{d\hat{T}}{dK} &= \frac{1}{b - a\hat{c}_1}\hat{T}w'(\bar{c}_2), \end{aligned} \tag{A.50}$$

and hence, since $b - a\hat{c}_1 > 0$ we have $dg/dK > 0$ and $d\hat{T}/dK < 0$. Thus, since $\frac{1}{\hat{T}} = \bar{\chi}(\hat{c}_1)$ at K, it follows that $\frac{1}{\hat{T}} > \bar{\chi}(\hat{c}_1)$ for K' larger than K but close to K. Hence, households hold only new coins. Also, since g increases, there is a lord type $\theta' > \bar{\theta}$ that chooses g. Then since $\hat{g}^{p*}(K') \geq g$ this implies that $\hat{g}^{p*}(K') > \hat{g}^{p*}(K)$ and, since $\partial^2 z/\partial g \partial\theta > 0$, the cutoff value for θ in (37) increases as well.[29] Thus, if $C^d > C^p$, the set of lord preference parameters θ that results in an optimal choice of a system of periodic re-coinage becomes larger when the cost of the non-cash alternative increases. ■

[29] At the old cutoff $\bar{\theta}$, we now have $z\left(\hat{g}^{p*}(K'), \bar{\theta}\right) - C^p > z\left(\hat{g}^{p*}(K), \bar{\theta}\right) - C^p = z\left(g^{d*}\left(\bar{\theta}\right), \bar{\theta}\right) - C^d$. Since $\partial^2 z/\partial g \partial\theta > 0$ and hence $\frac{\partial z(\hat{g}^{p*},\theta)}{\partial\theta} < \frac{dz\left(g^{d*}(\theta),\theta\right)}{d\theta} = \frac{\partial z\left(g^{d*}(\theta),\theta\right)}{\partial\theta}$ for $\theta > \hat{\theta}$ it follows that the cutoff $\bar{\theta}$ increases.

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