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# Long-Term Relationship Bargaining*

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**Abstract**

We analyze a bargaining model where there is a long-term relationship between a seller and a buyer and there is bargaining over a sequence of surpluses that arrives at fixed points in time. Markov Perfect Equilibria are analyzed and equilibrium payoffs characterized. The transfers between the players can be described as a first-order system of difference equations. Payoffs depend on both current and future surpluses. Future surpluses are important partly because the risk of separation leads to the loss of surplus today and in the future and partly because delay without separation can last into future periods. We also find conditions for existence and uniqueness of equilibria with immediate agreement.

**Keywords :** Bargaining, long term relationship.

**JEL classification:** C72, C78.

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# 1 Introduction

Negotiations often take place in long-term relationships where surpluses sequentially arrive over time. Some examples of such situations are firm-worker bargaining in models where labor is not perfectly mobile and bargaining between upstream and downstream firms when there is a long-term relationship between the two firms. This problem has also received attention in the literature. Muthoo (1995) analyzes a model where players bargain over a sequence of surpluses and where the arrival of future surpluses depends on the time of agreement. Also, the papers by Felli and Harris (1996) and Leach (1997) analyze models where the sequence of bargaining situations are interdependent. More recently, Hall and Milgrom (2008) analyzed a bargaining model with both a probability of breakdown and conflicts, following the lines of Binmore, Rubinstein, and Wolinsky (1986). There is also an extensive macro literature on repeated wage bargaining; see e.g., Christiano, Trabandt, and Walentin (2011).

In this paper, we analyze a bargaining model when there is a long-term relationship between a buyer and a seller and when there is bargaining over a sequence of surpluses. We analyze Markov Perfect Equilibria and characterize the equilibrium payoffs.

In the model, a seller and buyer are locked into a long-term relationship. Surpluses arrive sequentially at given points in time. When a surplus arrives, the seller and buyer bargain over the surplus for a fixed number of rounds, whereafter the surplus vanishes. Initially, when the surplus arrives, either the buyer or seller is randomly selected to be the proposer. If the proposal is accepted, the surplus is divided according to the proposal and if the proposal is rejected, play either moves to the next round or breaks down. If play moves to the next round, players get disagreement payoffs that are different from the breakdown payoffs. In the next round, a proposer is randomly selected to make a proposal and so on. Bargaining over the surplus thus proceeds for a given number of rounds until the surplus is forfeited after the final round. A new surplus arrives and then bargaining over the surplus starts anew. The model generalizes the credible bargaining framework in Hall and Milgrom (2008) that also allows for both a probability of breakdown during negotiations and an outside option based on the payoffs under disagreement. In addition we allow for an arbitrary number of bargaining rounds in each time period and a more general payoff structure. We find conditions for the existence and uniqueness of immediate-agreement equilibrium.

We analyze the equilibrium payoffs and find that the payments between the players can be described as a first-order system of difference equations. The solution of this system is described as the number of rounds goes to infinity and equilibrium payoffs can be described in terms of (initial round) current surpluses and future values in the game besides payoffs in terms of disagreement and breakdown. Specifically, note that the transfers between the seller and the buyer depend both on current and future surpluses, despite the fact that parties only bargain over current surpluses; as soon as a new surplus arrive, bargaining over that surplus starts anew. This is partly because bargaining under a risk of breakdown entails the risk of losing both current ant future surpluses and partly because delay without separation can last until a new surplus arrives, which means that future surpluses affect the current bargaining outcome.

The model in this paper is different from the model in Muthoo (1995), where the time of arrival of future surpluses is dependent on the time of agreement. Nevertheless, the payoff is dependent on the expected value in future agreements. As in Muthoo (1995), the division of surplus is different from the division of the surplus in a standard bargaining game, see for example Rubinstein (1982), although for other reasons than in Muthoo (1995).

The bargaining model is introduced in section 2. Section 3 analyzes the equilibrium and finally section 4 concludes the paper.

# 2 The model

There are two parties $i \in N = \{b, s\}$ bargaining over a sequence of values $\{v_t\}_0^\infty$ where $v_t = \{v_t^b, v_t^s\} \in \mathbb{R}^2$ is the value that arrives in time period $t$. In a given time period $t$, the parties bargain over how to share the surplus. Bargaining in period $t$ over $v_t$ takes place in $R$ rounds; a round lasts for $\Delta = \frac{1}{R}$ units of time. If an agreement is reached between the parties on some payoff division, the distribution is implemented in round $r$ in period $t$. If no agreement is reached before or in round $R$ the surplus is forfeit. During bargaining, the parties separate exogenously with probability $\bar{\delta}$ when a proposal is rejected. Parties discount future values by the discount factor $\beta$ per time period. Thus, the value in period $t$ of receiving a unit of goods in period $t' > t$ is $\beta^{t'-t}$. In each round, the proposer is randomly selected with probability $p_t^s$ for the seller and $p_t^b$ for the buyer. A strategy in the game for player $i$ is denoted $\sigma_i$. Let $\sigma = (\sigma_i)_{i \in N}$ and let $\Sigma$ denote the set of strategy profiles. In general, the strategy at any round in time period $t$ is a function the history up to that round.

Payoffs are potentially nonlinearly dependent on $v_t$. Let $\Upsilon_t^r(v_t^s)$ denote the current (net transfer) payoff of the seller in round $r$ and time period $t$ if agreement is reached in that round and time period. Similarly, let $\phi_t^r(v_t^b)$ denote the current (net transfer) payoff of the buyer in round $r$ and time period $t$. As an example, suppose a firm is bargaining with a worker over a fixed labor input of the worker that is used in the production of a good that the firm sells. $\Upsilon_t^r$ is then the utility cost of supplying labor for the worker and $\phi_t^r$ the gross profits of the firm. We assume that surpluses are nonincreasing in rounds; $\Upsilon_t^{r'} \leq \Upsilon_t^r$ and

$\phi_t^{r'} \geq \phi_t^r$ for $r' > r$. An example of $\Upsilon_t^r$ and $\phi_t^r$ that satisfies this is as follows. Assume there is a fixed surplus $v_t$ that arrives at the start of $t$ and that shrinks by $\theta$; we have $\Upsilon_t^r = \theta^{(r-1)\Delta} v_t^s$ and $\phi_t^r = \theta^{(r-1)\Delta} v_t^b$. The net surplus of agreement is $\phi_t^r + \Upsilon_t^r$. Letting $W_t^r$ denote the transfer between agents in cases where an agreement is reached, total current payoffs are then $\Upsilon_t^r + W_t^r$ and $\phi_t^r - W_t^r$. Furthermore, let $H$ denote the set of histories and let $h_t^r$ denote the history up to round $r$ and time $t$ and let $h_t^0$ denote the history before the first round in period $t$. For any strategy profile $\sigma \in \Sigma$, let $\sigma\left(h_{t+1}^r\right)$ denote the restriction of $\sigma$ to the histories consistent with $h_{t+1}^r$. Given the history $h_{t+1}^r$, let $V_{t+1}^r\left(h_{t+1}^r, \sigma\left(h_{t+1}^r\right)\right)$ denote the present value of the seller that accrues if play follows the strategy profile $\sigma\left(h_{t+1}^r\right)$ following $h_{t+1}^r$ for all periods $t' \geq t+1$. Similarly, let $F_{t+1}^r\left(h_{t+1}^r, \sigma\left(h_{t+1}^r\right)\right)$ denote the present value of the seller. Given a strategy profile $\sigma$ that prescribes agreement in round $r$ and period $t$ on the transfer $W_t^r$, the continuation payoff of the seller is

$$\Upsilon_t^r + W_t^r + \beta V_{t+1}^0\left(h_{t+1}^0, \sigma\left(h_{t+1}^0\right)\right) \tag{1}$$

and the continuation payoff of the buyer is

$$\phi_t^r - W_t^r + \beta F_{t+1}^0\left(h_{t+1}^0, \sigma\left(h_{t+1}^0\right)\right), \tag{2}$$

where $h_{t+1}^0 = \{h_t^r, W_t^r, A\}$. The model is a generalization of Hall and Milgrom (2008), both because we allow for $R > 1$ rounds and for a more general payoff structure. Thus, let $U_t$ denote the value for the seller when there is a breakdown in bargaining. For the buyer, the value in case of a breakdown is normalized to zero. If an agreement is not reached in a bargaining round $r$, the seller receives $\hat{z}_t\Delta$ and the buyer $\hat{\gamma}_t\Delta$ in the round. We assume $\hat{z}_t > 0$ and $\hat{\gamma}_t > 0$. The difference between this model and the model in Muthoo (1995) is that surpluses arrive at fixed points in time in this paper while they arrive at a fixed time after an agreement in Muthoo (1995). Moreover, proposers are selected at random in this paper instead of sequentially.

In the paper, we focus on Markov strategies. A Markov strategy depends on $r$, $t$ and the payoff relevant variables. A Markov Perfect Equilibrium is a SPE in Markov strategies (MPE). For a formal treatment, see Maskin and Tirole (2001).

# 3 Equilibrium

By standard arguments, in any MPE where an offer is accepted with positive probability, when being selected as proposer, the proposer offers the respondent a transfer such that the respondent is indifferent between accepting and rejecting. Let $W_t^{r,s}$ denote a proposal by the seller and let $W_t^{r,b}$ denote a proposal by the buyer in round $r$. Define $\Delta = \frac{1}{R}$ and let $\delta = \frac{\bar{\delta}}{\Delta}$. For now, we restrict attention to equilibria where an agreement is reached; below we describe conditions for existence and uniqueness. Then, as long as $r < R$, in any equilibrium prescribing an agreement, the buyer offers the seller $W_t^{r,b}$ such that

$$\begin{aligned} & \Upsilon_t^r + W_t^{r,b} + \beta^{\frac{R-r+1}{R}} V_{t+1}^0 \left(h_{t+1}^0, \sigma\left(h_{t+1}^0\right)\right) \\ = \; & \delta\Delta U_t + (1-\delta\Delta)\, p_t^b \left[\hat{z}_t \Delta + \beta^{\frac{1}{R}} \left(\Upsilon_t^{r+1} + W_t^{r+1,b} + \beta^{\frac{R-r}{R}} V_{t+1}^0 \left(h_{t+1}^{0\prime}, \sigma\left(h_{t+1}^{0\prime}\right)\right)\right)\right] \\ & + (1-\delta\Delta)\, p_t^s \left[\hat{z}_t \Delta + \beta^{\frac{1}{R}} \left(\Upsilon_t^{r+1} + W_t^{r+1,s} + \beta^{\frac{R-r}{R}} V_{t+1}^0 \left(h_{t+1}^{0\prime\prime}, \sigma\left(h_{t+1}^{0\prime\prime}\right)\right)\right)\right], \end{aligned} \tag{3}$$

where $h_{t+1}^0$, $h_{t+1}^{0\prime}$ and $h_{t+1}^{0\prime\prime}$ are identical up to period $t$ and before round $r$. Following round $r$ in period $t$, $h_{t+1}^0$ prescribes acceptance of $W_t^{r,b}$ while $h_{t+1}^{0\prime}$ and $h_{t+1}^{0\prime\prime}$ prescribes rejection of $W_t^{r,b}$ and acceptance of $W_t^{r+1,b}$ and $W_t^{r+1,s}$, respectively. Note that, since we analyze Markov Perfect equilibria, we have $V_{t+1}^0\left(h_{t+1}^0, \sigma\left(h_{t+1}^0\right)\right) = V_{t+1}^0\left(h_{t+1}^{0\prime}, \sigma\left(h_{t+1}^{0\prime}\right)\right) = V_{t+1}^0\left(h_{t+1}^{0\prime\prime}, \sigma\left(h_{t+1}^{0\prime\prime}\right)\right) = V_{t+1}^0$ and similarly for the seller. Then, as long as $r < R$, the seller offers the buyer $W_t^{r,s}$ such that

$$\begin{aligned} & \phi_t^r - W_t^{r,s} + \beta^{\frac{R-r+1}{R}} F_{t+1}^0 = (1-\delta\Delta)\, p_t^b \left[\hat{\gamma}_t \Delta + \beta^{\frac{1}{R}} \left(\phi_t^{r+1} - W_t^{r+1,b} + \beta^{\frac{R-r}{R}} F_{t+1}^0\right)\right] \\ & + (1-\delta\Delta)\, p_t^s \left[\hat{\gamma}_t \Delta + \beta^{\frac{1}{R}} \left(\phi_t^{r+1} - W_t^{r+1,s} + \beta^{\frac{R-r}{R}} F_{t+1}^0\right)\right]. \end{aligned} \tag{4}$$

When $r = R$, the values are

$$\begin{aligned} & \Upsilon_t^R + W_t^{R,b} + \beta^{\frac{1}{R}} V_{t+1}^0 = \delta\Delta U_t + (1-\delta\Delta)\, p_{t+1}^b \left[\hat{z}_t \Delta + \beta^{\frac{1}{R}} \left(\Upsilon_{t+1}^0 + W_{t+1}^{0,b} + \beta V_{t+2}^0\right)\right] \\ & + (1-\delta\Delta)\, p_{t+1}^s \left[\hat{z}_t \Delta + \beta^{\frac{1}{R}} \left(\Upsilon_{t+1}^0 + W_{t+1}^{0,s} + \beta V_{t+2}^0\right)\right] \end{aligned} \tag{5}$$

and

$$\begin{aligned} & \phi_t^R - W_t^{R,s} + \beta^{\frac{1}{R}} F_{t+1}^0 = (1-\delta\Delta)\, p_{t+1}^b \left[\hat{\gamma}_t \Delta + \beta^{\frac{1}{R}} \left(\phi_{t+1}^0 - W_{t+1}^{0,b} + \beta F_{t+2}^0\right)\right] \\ & + (1-\delta\Delta)\, p_{t+1}^s \left[\hat{\gamma}_t \Delta + \beta^{\frac{1}{R}} \left(\phi_{t+1}^0 - W_{t+1}^{0,s} + \beta F_{t+2}^0\right)\right]. \end{aligned} \tag{6}$$

Before proving the main result, the following example illustrates that transfers in a given period $t$ depend on e.g., surpluses in all time periods following $t$.

**Example 1** *A simple example with two time periods and one round in each time period. Proposer prob-*

*abilities are* $\frac{1}{2}$, $\beta = 1$ *and* $\Upsilon_t^1 = 0$ *for* $t = \{1, 2\}$, $U_t = U$, $\hat{z}_1 = \hat{z}_2 = \hat{z}$, $\hat{\gamma}_1 = \hat{\gamma}_2 = \hat{\gamma}$, *for* $t = \{1, 2\}$ *and* $\phi_1^1 \neq \phi_2^1$. *In the last round, we have*

$$\begin{aligned} W_2^{1,s} &= \phi_2^1 - (1 - \delta\Delta)\,\hat{\gamma}\Delta \\ W_2^{1,b} &= \delta\Delta U + (1 - \delta\Delta)\,\hat{z}\Delta. \end{aligned} \tag{7}$$

*Then the continuation payoffs at the beginning of period* 2 *can be written as, using* $\Delta = 1$,

$$\begin{aligned} F_2^0 &= \frac{1}{2}(1-\delta)\,\hat{\gamma} + \frac{1}{2}\left(\phi_2^1 - (\delta U + (1-\delta)\,\hat{z})\right) \\ V_2^0 &= \frac{1}{2}\left(\phi_2^1 - (1-\delta)\,\hat{\gamma}\right) + \frac{1}{2}\left(\delta U + (1-\delta)\,\hat{z}\right). \end{aligned} \tag{8}$$

*In the first time period transfers are, using (5) and (6),*

$$\begin{aligned} W_1^{1,s} &= \phi_1^1 + F_2^0 - (1-\delta)\frac{1}{2}\left[\hat{\gamma} + \left(\phi_2^1 - W_2^{1,b}\right)\right] - (1-\delta)\frac{1}{2}\left[\hat{\gamma} + \left(\phi_2^1 - W_2^{1,s}\right)\right], \\ & W_1^{1,b} = \delta U + (1-\delta)\frac{1}{2}\left[\hat{z} + W_2^{1,b}\right] + (1-\delta)\frac{1}{2}\left[\hat{z} + W_2^{1,s}\right] - V_2^0. \end{aligned} \tag{9}$$

*Clearly, the wage* $W_1^{1,s}$ *and* $W_1^{1,b}$ *depend on second period surplus, as long as* $\delta < 1$.

Note that the transfers between the seller and the buyer depend both on current and future surpluses, despite the fact that parties start to bargain as soon as a new surplus arrives, as long as breakdown probability is less than one. This is partly because a breakdown of negotiations risks losing both current and future surpluses, in turn having the implication that future surpluses affect the current bargaining outcome, and partly because delay without separation might last until future surpluses arrive.

We can rearrange the equilibrium conditions for making an acceptable proposal (3) and (4) in any round $r < R$ so that the transfers between the seller and the buyer can be characterized by two difference equations. Thus, we have, for $r \leq R - 1$,

$$\begin{pmatrix} W_t^{r,b} \\ W_t^{r,s} \end{pmatrix} = (1 - \delta\Delta)\,\beta^{\frac{1}{R}}\bar{A}_t \begin{pmatrix} W_t^{r+1,b} \\ W_t^{r+1,s} \end{pmatrix} + B_t^r \tag{10}$$

where

$$\bar{A}_t = \begin{pmatrix} p_t^b & 1 - p_t^b \\ p_t^b & 1 - p_t^b \end{pmatrix} \tag{11}$$

and

$$B_t^r = \left( \begin{array}{c} \left((1-\delta\Delta)\left(\hat{z}_t\Delta + \left[\beta^{\frac{1}{R}}\Upsilon_t^{r+1} + \beta^{\frac{R-r+1}{R}}V_{t+1}^0 - U_t\right]\right) - \left(\Upsilon_t^r + \beta^{\frac{R-r+1}{R}}V_{t+1}^0 - U_t\right)\right) \\ -\left[(1-\delta\Delta)\left(\hat{\gamma}_t\Delta + \left[\beta^{\frac{1}{R}}\phi_t^{r+1} + \beta^{\frac{R-r+1}{R}}F_{t+1}^0\right]\right) - \left(\phi_t^r + \beta^{\frac{R-r+1}{R}}F_{t+1}^0\right)\right] \end{array} \right). \tag{12}$$

This follows easily by rearranging expressions (3) and (4). Intuitively, the current transfers are equal to a combination of a current round payoffs, as given by $B_t^r$, plus a probability-weighted average of the transfers in the next round, modified by discounting and the probability of breakdown. Note that $\bar{A}_t$ is idempotent.

Moreover, the equilibrium conditions (5) and (6) for making an acceptable proposal in the last round of period $t$, i.e., when a rejection leads to complete forfeit of the surplus in period $t$ and continued bargaining in period $t+1$, can be rewritten as

$$\begin{pmatrix} W_t^{R,b} \\ W_t^{R,s} \end{pmatrix} = (1-\delta\Delta)\,\beta^{\frac{1}{R}}\bar{A}_{t+1}\begin{pmatrix} W_{t+1}^{1,b} \\ W_{t+1}^{1,s} \end{pmatrix} + B_t^R \tag{13}$$

where

$$B_t^R = \left( \begin{array}{c} \left((1-\delta\Delta)\left(\hat{z}_t\Delta + \left[\beta^{\frac{1}{R}}V_{t+1}^0 - U_t\right]\right) - \left(\Upsilon_t^R + \beta^{\frac{1}{R}}V_{t+1}^0 - U_t\right)\right) \\ -\left[(1-\delta\Delta)\left(\hat{\gamma}_t\Delta + \beta^{\frac{1}{R}}F_{t+1}^0\right) - \left(\phi_t^R + \beta^{\frac{1}{R}}F_{t+1}^0\right)\right] \end{array} \right). \tag{14}$$

By repeatedly using expressions (10) - (12) and expressions (13) - (14) in a MPE with immediate agreement, the solution for the transfers between the seller and the buyer is given by

$$\begin{pmatrix} W_t^{1,b} \\ W_t^{1,s} \end{pmatrix} = (1-\delta\Delta)^{R-1}\,\beta^{\frac{R-1}{R}}\bar{A}_t\begin{pmatrix} W_t^{R,b} \\ W_t^{R,s} \end{pmatrix} + B_t^1 + \sum_{r=2}^{R-1}(1-\delta\Delta)^{r-1}\,\beta^{\frac{r-1}{R}}\bar{A}_t B_t^r \tag{15}$$

and

$$\begin{pmatrix} W_t^{R,b} \\ W_t^{R,s} \end{pmatrix} = (1-\delta\Delta)\,\beta^{\frac{1}{R}}\bar{A}_{t+1}\begin{pmatrix} W_{t+1}^{1,b} \\ W_{t+1}^{1,s} \end{pmatrix} + B_t^R. \tag{16}$$

The following Lemma describes the transfers as a first-order difference equation.

**Lemma 1** *The transfers between the seller and the buyer are given by the following difference equation:*

$$\begin{pmatrix} W_t^{1,b} \\ W_t^{1,s} \end{pmatrix} = (1-\delta\Delta)^{R}\,\beta\bar{A}_{t+1}\begin{pmatrix} W_{t+1}^{1,b} \\ W_{t+1}^{1,s} \end{pmatrix} + B_t^1 + \sum_{r=2}^{R}(1-\delta\Delta)^{r-1}\,\beta^{\frac{r-1}{R}}\bar{A}_t B_t^r. \tag{17}$$

The following example shows that uniqueness of an immediate agreement equilibrium cannot be guaranteed.

**Example 2** *An example with two time periods and one round in each time period (i.e., $\Delta = 1$). Proposer probabilities are $\frac{1}{2}$, $\beta = 1$, $\hat{z}_t = \hat{\gamma}_t = 0$ and $\Upsilon_{1,t} = 0$ for $t = \{1, 2\}$, $U_1 = \phi_2^1 + \gamma$ and $\phi_2^1 > U_2$. In the last round, equilibrium prescribes agreement and we have*

$$\begin{aligned} W_2^{1,s} &= \phi_2^1 \\ W_2^{1,b} &= \delta U_2. \end{aligned} \tag{18}$$

*Then the continuation payoffs at the beginning of period 2 can be written as*

$$\begin{aligned} F_2^0 &= \frac{1}{2}\left(\phi_2^1 - \delta U_2\right) \\ V_2^0 &= \frac{1}{2}\left(\phi_2^1 - \delta U_2\right) + \delta U_2. \end{aligned} \tag{19}$$

*Consider a candidate equilibrium where unacceptable offers are made in period 1. Then payoffs in period 1 are*

$$\begin{aligned} &(1-\delta) F_2^0 \\ &\delta \Delta U_2 + (1-\delta) V_2^0. \end{aligned} \tag{20}$$

*If the buyer deviates and makes an acceptable offer $\hat{W} = W_1^{1,b} + \varepsilon$ for some $\varepsilon > 0$, the wage has to satisfy*

$$\left(\hat{W} - \varepsilon\right) + V_2^0 = \delta \Delta U_1 + (1-\delta) V_2^0 \Rightarrow \hat{W} = \delta U_2 - \delta V_2^0 + \varepsilon. \tag{21}$$

*Then the gain from making an acceptable offer is, using that $F_2^0 + V_2^0 = \phi_2^1$,*

$$\phi_1^1 + \delta\left(\phi_2^1 - U_1\right) - \varepsilon. \tag{22}$$

*The above expression is violated if*

$$\frac{\phi_1^1}{\delta} < U_1 - \phi_2^1 = \gamma. \tag{23}$$

*Since an identical condition can be established if the seller deviates and makes an acceptable offer, there is an equilibrium with zero probability of agreement in period 1 if $\gamma$ is large enough. Note that the above* *expression implies* $\phi_1^1 + \phi_2^1 < U_1$*, since* $\delta < 1$ *when* $R = 1$*.*

To ensure existence, we impose the following conditions on payoffs.

**Condition 1** *For all* $t$ *and* $r < R$*,*

$$\phi_t^r + \Upsilon_t^r > \phi_t^{r+1} + \Upsilon_t^{r+1} + \hat{z}_t\Delta + \hat{\gamma}_t\Delta$$

*and, for* $r = R$*,*

$$\phi_t^r + \Upsilon_t^r > \hat{z}_t\Delta + \hat{\gamma}_t\Delta.$$

**Condition 2** *For all* $r, t$ *we have*

$$\phi_t^r + \Upsilon_t^r + \beta^{\frac{R-r+1}{R}} \sum_{i=1}^{\infty} \left(\beta^{i-1}\right) \left(\phi_{t+i}^1 + \Upsilon_{t+i}^1\right) > U_t.$$

The first condition requires that there is a surplus in agreeing rather than disagreeing and remaining in the relationship and the second that there is a surplus in disagreeing rather than separating.

**Proposition 1** *If conditions 1 and 2 are satisfied, an immediate agreement MPE exists for any* $\Delta \leq 1$*.*

**Proof**: See the appendix. ■

To show uniqueness we restrict attention to convergent sequences of $U_t$, $\hat{z}_t$ and $\hat{\gamma}_t$. Let

$$\begin{aligned} \bar{U} &= \lim_{t\to\infty} U_t \\ \bar{z} &= \lim_{t\to\infty} \hat{z}_t \\ \bar{\gamma} &= \lim_{t\to\infty} \hat{\gamma}_t. \end{aligned} \tag{24}$$

We restrict the limits of the inside and outside options in the following way.

**Condition 3** $\bar{U}$*,* $\bar{z}$ *and* $\bar{\gamma}$ *satisfies*

$$\bar{U} < \bar{z} + \bar{\gamma}.$$

**Proposition 2** *If conditions 1, 2 and 3 are satisfied, then there is a* $\bar{\beta}$ *such that, for any* $\beta > \bar{\beta}$*,*[1] *the immediate agreement MPE is unique for any* $\Delta < \hat{\Delta}$*.*

[1] *Note that* $\bar{\beta} = \frac{1}{e}$*. If the yearly discount rate is 4%, the condition is satisfied if surpluses arrive in intervals of up to almost 25 years.*

**Proof**: See the appendix. ■

Due to conditions 1 and 2, the only reason for delaying is that the value of breakdown in the future is larger than remaining in the relationship. Given the fairly mild condition 3, the proposition rules out the case with delayed agreement. [2]

Equilibrium transfers as $\Delta \to 0$ can be found by repeatedly using expressions (15) - )16), together with the following continuity and boundedness conditions.

**Condition 4** *For all $r, t$ such that $r < R-1$ we have*

$$\begin{aligned} \lim_{\Delta \to 0} \phi_t^{r+1} &= \phi_t^r \\ \lim_{\Delta \to 0} \Upsilon_t^{r+1} &= \Upsilon_t^r. \end{aligned}$$

*Furthermore,* $\lim_{\Delta \to 0} \phi_t^R = 0$ *and* $\lim_{\Delta \to 0} \Upsilon_t^R = 0$.

**Condition 5** *The sequence of surpluses* $\{\Upsilon_t^1\}_{t=1}^{\infty}$ *and* $\{\phi_t^1\}_{t=1}^{\infty}$ *satisfies*

$$\lim_{t \to \infty} \lim_{\Delta \to 0} \sum_{i=1}^{\infty} \left( \prod_{j=1}^{t+i} \left( (1-\delta\Delta)^R \right) \beta \right) \left( \phi_{t+i}^1 + \Upsilon_{t+i}^1 \right) = 0. \tag{25}$$

Thus, the surpluses grow slower than breakdown-adjusted discounting, ensuring that the total discounted value when bargaining is finite.

The equilibrium transfers are given by the following proposition.

**Proposition 3** *Suppose conditions 4 and 5 are satisfied. In a MPE with immediate agreement, the solution to the system of difference equations (15) and (16) when $\Delta \to 0$ is given by*

$$\begin{pmatrix} W_t^{1,b} \\ W_t^{1,s} \end{pmatrix} = \sum_{i=0}^{\infty} \beta^i e^{-\delta i} \bar{A}_{t+i} \left( \bar{B}_{t+i} + \bar{D}_{t+i} \right), \tag{26}$$

[2] If the sequences are not convergent, there is a unique equilibrium if condition 3 is modified to

$$\frac{\delta}{\delta - \ln \beta} \lim \inf_{t \to \infty} U_t + \frac{-\ln \beta}{\delta - \ln \beta} \lim \inf_{t \to \infty} \left( \hat{z}_t + \hat{\gamma}_t \right) > \lim \sup_{t \to \infty} U_t$$

*where*

$$\bar{B}_{t+i} = \frac{1-\beta e^{-\delta}}{\delta - \ln\beta}\begin{pmatrix} \delta U_{t+i} + \hat{z}_{t+i} - \delta V^0_{t+i+1} \\ -\hat{\gamma}_{t+i} + \delta F^0_{t+i+1} \end{pmatrix} \tag{27}$$

$$\bar{D}_{t+i} = \begin{pmatrix} -\Upsilon^1_{t+i} \\ \phi^1_{t+i} \end{pmatrix}. \tag{28}$$

*Moreover,*

$$\lim_{\Delta\to 0} W_t^{1,b} = \lim_{\Delta\to 0} W_t^{1,s}.$$

**Proof**: **Step** 1. Preliminaries and showing $\lim_{\Delta\to 0} W_t^{1,b} = \lim_{\Delta\to 0} W_t^{1,s}$.

Note that we can write

$$B_t^r = \bar{B}_t^r + \bar{D}_t^r, \tag{29}$$

where, for $r < R$,

$$\bar{B}_t^r = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Delta \begin{pmatrix} \delta U_t + (1-\delta\Delta)\,\hat{z}_t - \delta\left(\beta^{1-r\Delta+\Delta} V^0_{t+1}\right) \\ (1-\delta\Delta)\,\hat{\gamma}_t - \delta\left(\beta^{1-r\Delta+\Delta} F^0_{t+1}\right) \end{pmatrix}, \tag{30}$$

and

$$\bar{D}_t^r = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \left(\left[\beta^{\Delta}(1-\delta\Delta)\,\Upsilon_t^{r+1} + \beta^{1-r\Delta+\Delta} V^0_{t+1}\right]\right) - \left(\Upsilon_t^r + \beta^{1-r\Delta+\Delta} V^0_{t+1}\right) \\ \left(\beta^{\Delta}(1-\delta\Delta)\,\phi_t^{r+1} + \beta^{1-r\Delta+\Delta} F^0_{t+1}\right) - \left(\phi_t^r + \beta^{1-r\Delta+\Delta} F^0_{t+1}\right) \end{pmatrix}. \tag{31}$$

and for $r = R$,

$$\bar{B}_t^R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \Delta \begin{pmatrix} \delta U_t + (1-\delta\Delta)\,\hat{z}_t - \delta\left(\left[\beta^{\Delta}\Upsilon^1_{t+1} + \beta^{1+\Delta} V^0_{t+2}\right]\right) \\ (1-\delta\Delta)\,\hat{\gamma}_t - \delta\left(\left[\beta^{\Delta}\phi^1_{t+1} + \beta^{1+\Delta} F^0_{t+2}\right]\right) \end{pmatrix} \tag{32}$$

and

$$\bar{D}_t^R = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} \left(\left[\beta^{\Delta} V^0_{t+1}\right]\right) - \left(\Upsilon_t^R + \beta^{\frac{1}{R}} V^0_{t+1}\right) \\ \left(\beta^{\Delta} F^0_{t+1}\right) - \left(\phi_t^R + \beta^{\frac{1}{R}} F^0_{t+1}\right) \end{pmatrix}. \tag{33}$$

Using expressions (15) - (16) we then have

$$\begin{aligned}\begin{pmatrix} W_t^{1,b} \\ W_t^{1,s} \end{pmatrix} &= \prod_{j=1}^{\hat{t}} \left( (1-\delta\Delta)^R \beta \bar{A}_{t+j} \right) \begin{pmatrix} W_{t+\hat{t}}^{1,b} \\ W_{t+\hat{t}}^{1,s} \end{pmatrix} \\ &+ \sum_{i=0}^{\hat{t}} \left( \prod_{j=1}^{i} \left( (1-\delta\Delta)^R \beta \bar{A}_{t+j} \right) \right) \left( B_{t+i}^1 + \sum_{r=2}^{R} (1-\delta\Delta)^{r-1} \beta^{\frac{r-1}{R}} \bar{A}_{t+i} B_{t+i}^r \right). \end{aligned} \tag{34}$$

If the sequences $\{\Upsilon_t^1\}_{t=1}^{\infty}$ and $\{\phi_t^1\}_{t=1}^{\infty}$ satisfy condition 5, we have

$$\lim_{\hat{t}\to\infty} \prod_{j=1}^{\hat{t}} \left( (1-\delta\Delta)^R \beta \bar{A}_{t+j} \right) \begin{pmatrix} W_{t+\hat{t}}^{1,b} \\ W_{t+\hat{t}}^{1,s} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}. \tag{35}$$

Hence the first term in expression (34) goes to zero as $\hat{t} \to \infty$. Moreover, noting that $B_t^r = \bar{B}_t^r + \bar{D}_t^r$ gives

$$\begin{aligned}\begin{pmatrix} W_t^{1,b} \\ W_t^{1,s} \end{pmatrix} &= \sum_{i=0}^{\infty} \left( \prod_{j=1}^{i} \left( (1-\delta\Delta)^R \beta \right) \bar{A}_{t+j} \right) \left( \bar{B}_{t+i}^1 + \bar{D}_{t+i}^1 \right) \\ &+ \sum_{i=0}^{\infty} \left( \prod_{j=1}^{i} \left( (1-\delta\Delta)^R \beta \right) \bar{A}_{t+j} \right) \left( \sum_{r=2}^{R-1} (1-\delta\Delta)^{r-1} \beta^{\frac{r-1}{R}} \bar{A}_{t+i} \left( \bar{B}_{t+i}^r + \bar{D}_{t+i}^r \right) \right) \\ &+ \sum_{i=0}^{\infty} \left( \prod_{j=1}^{i} \left( (1-\delta\Delta)^R \beta \right) \bar{A}_{t+j} \right) (1-\delta\Delta)^{R-1} \beta^{\frac{R-1}{R}} \bar{A}_{t+i} \left( \bar{B}_{t+i}^R + \bar{D}_{t+i}^R \right). \end{aligned} \tag{36}$$

Note that, by the definition of $\bar{A}_{t+i}$, the only terms that are different in $W_t^{1,b}$ and $W_t^{1,s}$ are $\bar{B}_t^1$ and $\bar{D}_t^1$. Then, since $\lim_{\Delta\to 0} \bar{B}_t^1 = 0$ and we have $\lim_{\Delta\to 0} \bar{D}_t^1 = 0$ by using condition 4, it follows that

$$\lim_{\Delta\to 0} W_t^{1,b} = \lim_{\Delta\to 0} W_t^{1,s}. \tag{37}$$

**Step** 2. Computing $\bar{B}_{t+i}$ and $\bar{D}_{t+i}$.

Define

$$\bar{B}_{t+i}^{\Sigma} = \sum_{r=1}^{R} (1-\delta\Delta)^{r-1} \beta^{\frac{r-1}{R}} \bar{B}_{t+i}^r. \tag{38}$$

Then, using the definition of $R$ and $\bar{B}_{t+i}^r$,

$$\bar{B}_{t+i}^{\Sigma} = \frac{1-(1-\delta\Delta)^R \beta^{\frac{R}{R}}}{\frac{1}{\Delta} - \frac{1}{\Delta}(1-\delta\Delta)\beta^{\frac{1}{R}}} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}^{-1} \begin{pmatrix} \delta U_t + (1-\delta\Delta)\hat{z}_t - \delta\left(\beta^{\frac{1}{R}} V_{t+1}^0\right) \\ (1-\delta\Delta)\hat{\gamma}_t - \delta\left(\beta^{\frac{1}{R}} F_{t+1}^0\right) \end{pmatrix}. \tag{39}$$

Letting

$$\bar{B}_{t+i} = \lim_{R\to\infty} \bar{B}^{\Sigma}_{t+i} \tag{40}$$

and using that

$$\lim_{R\to\infty} \frac{1-\left(1-\delta\frac{1}{R}\right)^{R}\beta}{R-R\left(1-\delta\frac{1}{R}\right)\beta^{\frac{1}{R}}} = \frac{1-\beta e^{-\delta}}{\delta-\ln\beta} \tag{41}$$

and the properties of $\bar{A}_{t+i}$, i.e., we have $\bar{A}_{t+j}\bar{A}_{t+i} = \bar{A}_{t+i}$, we can establish (27) in the proposition. To establish (28), let

$$\bar{D}^{\Sigma}_{t+i} = \sum_{r=2}^{R} (1-\delta\Delta)^{r-1}\beta^{\frac{r-1}{R}}\bar{D}^{r}_{t+i} \tag{42}$$

and note that $\bar{D}^{\Sigma}_{t+i}$ can be written as

$$\sum_{r=2}^{R-1} (1-\delta\Delta)^{r-1}\beta^{\frac{r-1}{R}} \begin{pmatrix} \beta^{\Delta}(1-\delta\Delta)\Upsilon^{r+1}_{t+i} - \Upsilon^{r}_{t+i} \\ \beta^{\Delta}(1-\delta\Delta)\phi^{r+1}_{t+i} - \phi^{r}_{t+i} \end{pmatrix} + (1-\delta\Delta)^{R-1}\beta^{\frac{R-1}{R}} \begin{pmatrix} -\Upsilon^{R}_{t+i} \\ -\phi^{R}_{t+i} \end{pmatrix}, \tag{43}$$

a telescoping series and hence, using that $\lim_{R\to\infty}\bar{D}^{1}_{t+i} = 0$, that $\bar{A}_{t+j}\bar{A}_{t+i} = \bar{A}_{t+i}$ and defining

$$\bar{D}_{t+i} = \lim_{R\to\infty}\left(\bar{D}^{1}_{t+i} + \bar{D}^{\Sigma}_{t+i}\right) = \begin{pmatrix} -\Upsilon^{1}_{t+i} \\ -\phi^{1}_{t+i} \end{pmatrix}$$

we can establish (28).■

The proof divides up the round payoff in (12) in terms of flow round payoffs in expression (27) and surplus changes between rounds in (28). Specifically, even if transfers in (15) depend on the surpluses in all rounds in the current period through $B^{r}_{t}$, the resulting wage depends only on the surplus in the first round in each time period. From a technical perspective, this is because the payoffs in expression (17) depend on the change in surpluses between rounds implying that surpluses for higher rounds enter payoffs in a telescoping way and hence cancel themselves out, leading to that transfers depend only on first-period surplus. Note also that, since agreement is reached in the first round, total payoffs depend only on first-period surplus. Thus, we can express the equilibrium payoffs partly in terms of future disagreement and separation payoffs, i.e., $U_{t+i}$, $\hat{z}_{t+i}$ and $\hat{\gamma}_{t+i}$, and partly in terms of future values of the problem, besides depending on current first round surplus.

**Remark 1** *Note that expression (17) in Lemma 1 can be written as*

$$W^{1,b}_{t} = e^{-\delta}\beta W^{1,b}_{t+1} + \bar{A}_{t}\left(\bar{B}_{t} + \bar{D}_{t}\right). \tag{44}$$

Generally these expressions are somewhat complicated. However, in the special case when the probability of breakdown vanishes (i.e., $\delta \to 0$), equilibrium payoffs have a simpler and more intuitive form. Then (26) becomes

$$\begin{pmatrix} W_t^{1,b} \\ W_t^{1,s} \end{pmatrix} = \sum_{i=0}^{\infty} \beta^i \bar{A}_{t+i} \begin{pmatrix} \frac{1-\beta}{-\ln\beta} \hat{z}_{t+i} - \Upsilon_{t+i}^1 \\ -\frac{1-\beta}{-\ln\beta} \hat{\gamma}_{t+i} + \phi_{t+i}^1 \end{pmatrix}. \tag{45}$$

**Remark 2** *Note that, if there is no discounting between rounds in a given time period, but rather only between time periods, the results above hold with minor modifications of conditions and results. Equation (3) is modified to*

$$\begin{aligned} & \Upsilon_t^r + W_t^{r,b} + \beta V_{t+1}^0 \\ = \; & \delta\Delta U_t + (1-\delta\Delta)\, p_t^b \left[\hat{z}_t \Delta + \left(\Upsilon_t^{r+1} + W_t^{r+1,b} + \beta V_{t+1}^0\right)\right] \\ & + (1-\delta\Delta)\, p_t^s \left[\hat{z}_t \Delta + \left(\breve{\Upsilon}_t^{r+1} + W_t^{r+1,s} + \beta V_{t+1}^0\right)\right] \end{aligned} \tag{46}$$

*Equations (4), (5) and (6) are modified similarly. Straightforward modifications of Propositions 1 and 2 establish existence and uniqueness. Proposition 3 is modified so that* $(1-\delta\Delta)^{r-1} \beta^{\frac{r-1}{R}}$ *is replaced by* $(1-\delta\Delta)^{r-1}$ *and the adjustment of* $B_t^r$ *in (12) is*

$$B_t^r = \begin{pmatrix} (1-\delta\Delta)\left(\hat{z}_t\Delta + \left[\Upsilon_t^{r+1} + \beta V_{t+1}^0 - U_t\right]\right) - \left(\Upsilon_t^r + \beta V_{t+1}^0 - U_t\right) \\ -\left[(1-\delta\Delta)\left(\hat{\gamma}_t\Delta + \left[\phi_t^{r+1} + \beta F_{t+1}^0\right]\right) - \left(\phi_t^r + \beta F_{t+1}^0\right)\right] \end{pmatrix}, \tag{47}$$

*which is reflected in the construction of* $\bar{B}_t^r$ *and* $\bar{D}_t^r$ *in the proof. In the statement of the proposition, expression (27) when* $\Delta \to 0$ *is modified to*

$$\bar{B}_{t+i} = \frac{1-e^{-\delta}}{\delta} \begin{pmatrix} \delta U_{t+i} + \hat{z}_{t+i} - \delta\beta V_{t+i+1}^0 \\ -\hat{\gamma}_{t+i} + \delta\beta F_{t+i+1}^0 \end{pmatrix}. \tag{48}$$

*Furthermore, we can write*

$$\begin{aligned} W_t^{1,b} \;=\; & \beta W_{t+1}^{1,b} + \frac{1-e^{-\delta}}{\delta}\left(\delta U_t + \hat{z}_t - \delta\beta V_{t+1}^0\right) - \Upsilon_t^1 \\ & + \left(1-p_t^b\right)\left[\frac{1-e^{-\delta}}{\delta}\left(-\hat{\gamma}_t + \delta\beta F_{t+1}^0 - \left(\delta U_t + \hat{z}_t - \delta\beta V_{t+1}^0\right)\right) + \left(\phi_t^1 + \Upsilon_t^1\right)\right] \end{aligned} \tag{49}$$

*As $\delta \to 0$ then $\lim_{\delta\to 0} \frac{1-e^{-\delta}}{\delta} = 1$ and hence*

$$\begin{aligned} W_t^{1,b} &= \delta\left(U_t - \beta V_{t+1}^0\right) + \hat{z}_t - \Upsilon_t^1 \\ &+ \left(1-p_t^b\right)\left(\phi_t^1 + \Upsilon_t^1 - \hat{\gamma}_t - \delta U_t - \hat{z}_t + \delta\beta\left(F_{t+1}^0 + V_{t+1}^0\right)\right) + \beta W_{t+1}^{1,b}. \end{aligned} \tag{50}$$

*Thus, the value of agreeing is equal to the disagreement value (the right-hand side terms on the first row) plus the seller proposer probability $(1 - p_t^b)$ times the surplus of agreeing plus the future wage (capturing future surpluses). When the probability of breakdown parameter $\delta$ goes to infinity, equilibrium payoffs depend only on the separation payoffs and not on $\hat{z}_t$ and $\hat{\gamma}_t$, besides current and future values. Too see this, note that, as $\delta \to \infty$, we get*

$$\begin{aligned} W_t^{1,b} &= \lim_{\delta\to\infty} p_t^b \left[\frac{1-e^{-\delta}}{\delta}\left(\delta U_t + \hat{z}_t - \delta\beta V_{t+1}^0\right) - \Upsilon_t^1\right] \\ &+ \lim_{\delta\to\infty}\left(1-p_t^b\right)\left[\frac{1-e^{-\delta}}{\delta}\left(-\hat{\gamma}_t + \delta\beta F_{t+1}^0\right) + \phi_t^1\right]. \end{aligned} \tag{51}$$

*Since $\lim_{\delta\to\infty} \frac{1-e^{-\delta}}{\delta} \to 0$, we thus get*

$$W_t^{1,b} = \left(1-p_t^b\right)\left(\phi_t^1 + \beta F_{t+1}^0\right) - p_t^b\left(\left(\Upsilon_t^1 + \beta V_{t+1}^0\right) - U_t\right), \tag{52}$$

*implying that the payoff of agreeing is*

$$\Upsilon_t^1 + W_t^{1,b} + \beta V_{t+1}^0 = \left(1-p_t^b\right)\left(\phi_t^1 + \beta F_{t+1}^0 + \left(\Upsilon_t^1 + \beta V_{t+1}^0\right) - U_t\right) + U_t, \tag{53}$$

*i.e., the separation outside option plus the proposal probability times the total value of agreeing. Note that the above expression differs from (45) in that the current payoff of agreeing for the seller depends on the outside option $U_t$.*

## 4 Concluding remarks

In the paper we analyze a bargaining model when there is a long-term relationship between a seller and a buyer and bargaining over a sequence of surpluses arriving at fixed points in time. Markov Perfect Equilibria are analyzed and the equilibrium payoffs characterized. We find conditions for uniqueness and existence of an immediate-agreement equilibrium. Furthermore, we show that the transfers between the players can be described as a first-order system of difference equations in terms of current and future transfers.

We find that the transfers between the seller and the buyer depend both on current and future surpluses, despite the fact that parties starts to bargain as soon as a new surplus arrives, as long as breakdown probability is positive. This is not only because bargaining under a risk of breakdown risks losing the surplus both today and in the future, but also because delay without separation might last until the arrival of future surpluses, which in turn means that future surpluses affect the current bargaining outcome.

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# Appendix

This appendix gives proofs of some of the results in the paper.

**Proof** of Proposition 1. Suppose the buyer is the proposer. The seller case follows by a similar argument.

**Case** 1**.** Suppose $r < R$. The wage when making an acceptable offer is

$$\begin{aligned} & \Upsilon_t^r + W_t^{r,b} + \beta^{\frac{R-r+1}{R}} V_{t+1}^0 \\ = \; & \delta\Delta U_t + (1-\delta\Delta)\, p_t^b \left[ \hat{z}_t\Delta + \beta^{\frac{1}{R}} \left( \Upsilon_t^{r+1} + W_t^{r+1b} + \beta^{\frac{R-r}{R}} V_{0,t+1}^0 \right)\right] \\ & + (1-\delta\Delta)\, p_t^s \left[ \hat{z}_t\Delta + \beta^{\frac{1}{R}} \left( \check{\Upsilon}_t^{r+1} + W_t^{r+1,s} + \beta^{\frac{R-r}{R}} V_{t+1}^0 \right)\right] \end{aligned} \tag{54}$$

and when making an unacceptable offer

$$(1-\delta\Delta)\left( -p_t^b \beta^{\frac{1}{R}} W_t^{r+1,b} - p_t^s \beta^{\frac{1}{R}} W_t^{r+1,s} + \left[ \hat{\gamma}_t\Delta + \left( \beta^{\frac{1}{R}} \phi_t^{r+1} + \beta^{\frac{R-r+1}{R}} F_{t+1}^0 \right)\right]\right). \tag{55}$$

The gain of making an acceptable offer is then

$$\phi_t^r - (1-\delta\Delta)\left( \beta^{\frac{1}{R}} \phi_t^{r+1} + \hat{\gamma}_t\Delta \right) - \left( W_t^{r,b} - (1-\delta\Delta)\, \beta^{\frac{1}{R}} \left( p_t^b W_t^{r+1,b} + p_t^s W_t^{r+1,s} \right)\right) + \delta\Delta\beta^{\frac{R-r+1}{R}} F_{t+1}^0. \tag{56}$$

From expression (10) - (12) we have

$$\begin{aligned} & \left( W_t^{r,b} - (1-\delta\Delta)\, \beta^{\frac{1}{R}} \left( p_t^b W_t^{r+1,b} + p_t^s W_t^{r+1,s} \right)\right) \\ = \; & \left( (1-\delta\Delta)\left( \hat{z}_t\Delta + \beta^{\frac{1}{R}} \Upsilon_t^{r+1} \right) - \left( \Upsilon_t^r + \delta\Delta \left( \beta^{\frac{R-r+1}{R}} V_{t+1}^0 - U_t \right)\right)\right) \end{aligned} \tag{57}$$

and hence expression (56) is

$$\phi_t^r + \Upsilon_t^r - (1-\delta\Delta)\left( \beta^{\frac{1}{R}} \phi_t^{r+1} + \beta^{\frac{1}{R}} \Upsilon_t^{r+1} + \hat{z}_t\Delta + \hat{\gamma}_t\Delta \right) + \delta\Delta\left( \beta\left( F_{t+1}^0 + V_{t+1}^0 \right) - U_t \right). \tag{58}$$

Using conditions (1), (2) and that

$$F_{t+1}^0 + V_{t+1}^0 = \sum_{i=1}^{\infty} \left( \beta^i \right) \left( \phi_{t+i}^1 + \Upsilon_{t+i}^1 \right)$$

we establish that there exists a $\breve{\Delta}$ such that

$$\begin{aligned} \phi_t^r + \Upsilon_t^r - \left(\beta^{\Delta}\phi_t^{r+1} + \beta^{\Delta}\Upsilon_t^{r+1} + \hat{z}_t\Delta + \hat{\gamma}_t\Delta\right) &> 0, \\ \phi_t^r + \Upsilon_t^r + \beta^{\frac{R-r+1}{R}}\left(F_{t+1}^0 + V_{t+1}^0\right) - U_t &> 0 \end{aligned} \tag{59}$$

for any $\Delta \leq 1$.

**Case** 2. Suppose $r = R$. The gain of making an acceptable offer is

$$\phi_t^R - (1-\delta\Delta)\,\hat{\gamma}_t\Delta - \left(W_t^{R,b} - (1-\delta\Delta)\,\beta^{\frac{1}{R}}\left(p_{t+1}^b W_{t+1}^{1,b} + p_{t+1}^s W_{t+1}^{1,s}\right)\right) + \delta\Delta\beta^{\frac{1}{R}}F_{t+1}^0. \tag{60}$$

From (16) we have

$$\begin{aligned} & \left(W_t^{R,b} - (1-\delta\Delta)\,\beta^{\frac{1}{R}}\left(p_{t+1}^b W_{t+1}^{1,b} + p_{t+1}^s W_{t+1}^{1,s}\right)\right) \\ = \; & \left((1-\delta\Delta)\left(\hat{z}_t\Delta + \left[\beta^{\frac{1}{R}}V_{t+1}^0 - U_t\right]\right) - \left(\Upsilon_t^R + \beta^{\frac{1}{R}}V_{t+1}^0 - U_t\right)\right) \end{aligned} \tag{61}$$

and hence (60) can be rewritten as

$$\phi_t^R + \Upsilon_t^R - (\hat{z}_t\Delta + \hat{\gamma}_t\Delta) + \delta\Delta\left[\beta^{\frac{1}{R}}\left(V_{t+1}^0 + F_{t+1}^0\right) + (\hat{z}_t\Delta + \hat{\gamma}_t\Delta) - U_t\right].$$

By similar arguments as in Case 1 there is a $\tilde{\Delta}$ such that the expression above is positive, for all $\Delta \leq 1$.

Combining case 1 and 2 and letting $\bar{\Delta} = \min\{\breve{\Delta}, \tilde{\Delta}\}$ *we* establish existence for any $\Delta \leq 1$. ■

**Proof** of Proposition 2. Suppose the buyer is the proposer. The seller case follows by a similar argument.

We first show that, if it is profitable to make an acceptable offer in round $r$ in period $t$, then it is profitable to make an acceptable offer in period $t-1$.

Suppose the MPE has a buyer proposal accepted in round $r \leq R$ for some $t$. Consider round $r-1$ and suppose the buyer proposes $\hat{W}$ such that

$$\begin{aligned} & \Upsilon_t^{r-1} + \hat{W} - \varepsilon + \beta^{\frac{R-r}{R}}V_{t+1}^0 \\ = \; & \delta\Delta U_t + (1-\delta\Delta)\,p_t^b\left[\hat{z}_t\Delta + \beta^{\frac{1}{R}}\left(\Upsilon_t^r + W_t^{r,b} + \beta^{\frac{R-r-1}{R}}V_{t+1}^0\right)\right] \\ & + (1-\delta\Delta)\,p_t^s\left[\hat{z}_t\Delta + \beta^{\frac{1}{R}}\left(\breve{\Upsilon}_t^{r+1} + W_t^{r,s} + \beta^{\frac{R-r-1}{R}}V_{t+1}^0\right)\right] \end{aligned} \tag{62}$$

for some $\varepsilon > 0$. Clearly, the seller accepts this offer with probability one for any $\varepsilon > 0$. The payoff when an unacceptable offer is made is

$$(1-\delta\Delta)\left(-p_t^b\beta^{\frac{1}{R}}W_t^{r,b}-p_t^s\beta^{\frac{1}{R}}W_t^{r,s}+\left[\hat{\gamma}_t\Delta+\left(\beta^{\frac{1}{R}}\phi_t^r+\beta^{\frac{R-r}{R}}F_{t+1}^0\right)\right]\right). \tag{63}$$

Note that we can set $\varepsilon$ such that $\hat{W}-\varepsilon=W_t^{r,b}$ and hence the gain of making an acceptable offer is then

$$\phi_t^{r-1}-(1-\delta\Delta)\left(\beta^{\frac{1}{R}}\phi_t^r+\hat{\gamma}_t\Delta\right)-\left(W_t^{r,b}+\varepsilon-(1-\delta\Delta)\,\beta^{\frac{1}{R}}\left(p_t^bW_t^{r,b}+p_t^sW_t^{r,s}\right)\right)+\delta\Delta\beta^{\frac{R-r}{R}}F_{t+1}^0. \tag{64}$$

Using expression (57) as in the proof of proposition 1 above, we have

$$\phi_t^{r-1}+\Upsilon_t^{r-1}-(1-\delta\Delta)\left(\beta^{\frac{1}{R}}\phi_t^r+\beta^{\frac{1}{R}}\Upsilon_t^r+\hat{z}_t\Delta+\hat{\gamma}_t\Delta\right)+\delta\Delta\left(\beta\left(F_{t+1}^0+V_{t+1}^0\right)-U_t\right)-\varepsilon. \tag{65}$$

For any $\beta>\frac{1}{e}\equiv\bar{\beta}$ we have

$$\frac{1}{\delta-\ln\beta}\left(\delta U_{t+v}+\hat{z}_{t+v}+\hat{\gamma}_{t+v}\right)>\frac{\delta}{\delta-\ln\beta}U_{t+v}+\frac{-\ln\beta}{\delta-\ln\beta}\left(\hat{z}_{t+v}+\hat{\gamma}_{t+v}\right). \tag{66}$$

Hence, from condition 3, there is a $t^*$ such that, for any $t>t^*$, we have

$$\begin{aligned}
&\frac{1-\beta e^{-\delta}}{\delta-\ln\beta}\sum_{v=1}^{\infty}\left(\beta e^{-\delta}\right)^{v-1}\left(\delta U_{t+v}+\hat{z}_{t+v}+\hat{\gamma}_{t+v}\right)\\
>\;&\left(1-\beta e^{-\delta}\right)\sum_{v=1}^{\infty}\left(\beta e^{-\delta}\right)^{v-1}\left(\frac{\delta}{\delta-\ln\beta}U_{t+v}+\frac{-\ln\beta}{\delta-\ln\beta}\left(\hat{z}_{t+v}+\hat{\gamma}_{t+v}\right)\right)>U_t.
\end{aligned} \tag{67}$$

Since

$$F_{t+1}^0+V_{t+1}^0\geq\frac{1-\beta e^{-\delta}}{\delta-\ln\beta}\sum_{v=1}^{\infty}\left(\beta e^{-\delta}\right)^{v-1}\left(\delta U_{t+v}+\hat{z}_{t+v}+\hat{\gamma}_{t+v}\right), \tag{68}$$

condition 1, 2 and 3 establishes that, for $\varepsilon$ small, there exists a $\check{\Delta}$ such that expression (65) is positive for $\Delta<\check{\Delta}$ whenever $\beta>\frac{1}{e}$.

Suppose the buyer proposal is not accepted in any round. Consider round $R$ and suppose the buyer proposes $\hat{W}$ such that

$$\Upsilon_t^R+\hat{W}-\varepsilon+\beta^{\frac{1}{R}}V_{t+1}^0=\delta\Delta U_t+(1-\delta\Delta)\,p_t^b\left[\hat{z}_t\Delta+\beta^{\frac{1}{R}}V_{t+1}^0\right]+(1-\delta\Delta)\,p_t^s\left[\hat{z}_t\Delta+\beta^{\frac{1}{R}}V_{t+1}^0\right]. \tag{69}$$

The seller accepts this offer with probability one as long as $\varepsilon>0$. The payoff when an unacceptable offer is made is

$$(1-\delta\Delta)\left[\hat{\gamma}_t\Delta+\beta^{\frac{1}{R}}F^0_{t+1}\right] \tag{70}$$

and hence the gain is

$$\phi^R_t+\Upsilon^R_t-(1-\delta\Delta)\left(\hat{z}_t\Delta+\hat{\gamma}_t\Delta\right)+\delta\Delta\left(\beta^{\frac{1}{R}}\left(V^0_{t+1}+F^0_{t+1}\right)-U_t\right)-\varepsilon. \tag{71}$$

Again, for $\varepsilon$ small, there exists a $\breve{\Delta}$ such that expression (65) is positive for $\Delta<\breve{\Delta}$ whenever $\beta>\frac{1}{e}$.

Combining case 1 and 2 and letting $\hat{\Delta}=\min\{\breve{\Delta},\check{\Delta}\}$ we establish uniqueness for any $\Delta<\hat{\Delta}$ whenever $\beta>\frac{1}{e}$. ■

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